101-1 微乙期末考試題及詳解

1. 求 $\lim_{x\to 0}\dfrac{\sin x - x\cos x}{\tan^{-1}x - \sin x}$.

   Sol.

   (1) 此題為羅比達法則應用

   (2) 已知 $\lim_{x\to 0}\dfrac{\sin}{x} = 1$

$$
\begin{aligned}
& \lim_{x\to 0}\frac{\sin x - x\cos x}{\tan^{-1}x - \sin x} && \left(\frac{0}{0}\right)\\
= & \lim_{x\to 0}\frac{\cos -(\cos x - x\sin x)}{\frac{1}{1+x^2} - \cos x} && \left(\frac{0}{0}\right)\\
= & \lim_{x\to 0}\frac{x\sin x}{\frac{1}{1+x^2} - \cos x} && \left(\frac{0}{0}\right)\\
= & \lim_{x\to 0}\frac{\sin x + x\cos x}{\frac{-2x}{(1+x^2)^2} + \sin x} && \left(\frac{0}{0}\right)\\
= & \lim_{x\to 0}\frac{\frac{\sin}{x} + \cos x}{\frac{-2}{(1+x^2)^2} + \frac{\sin x}{x}}\\
= & \frac{1+1}{-2+1} = -2
\end{aligned}
$$

□

2. 計算 $\int \frac{1}{x^2+3x+2}dx.$

Sol.

$$
\begin{aligned}
& \int \frac{1}{x^2+3x+2} \\
= & \int \frac{1}{(x+2)(x+1)}dx \\
= & \int \frac{-1}{x+2}dx + \int \frac{1}{x+1}dx \\
= & -\ln|x+2| + \ln|x+1| + C \\
= & \ln\left|\frac{x+1}{x+2}\right| + C
\end{aligned}
$$

□

3. 計算 $\int x^2 \sin x dx.$

Sol.

分部積分公式：

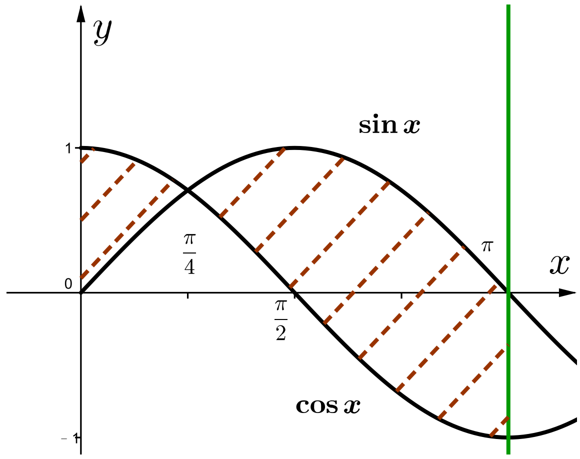
$$\int u dv = uv - \int v du.$$

$$
\begin{aligned}
& \int x^2 \sin x dx \\
& \left(令 \begin{array}{ll} u = x^2 & du = 2xdx \\ dv = \sin x dx & v = -\cos x \end{array} \right) \\
= & x^2 \sin x dx = -x^2 \cos x + \int 2x \cos x dx \\
& \left(令 \begin{array}{ll} u = 2x & du = 2dx \\ dv = \cos x dx & v = \sin x \end{array} \right) \\
= & -x^2 \cos x + (2x \sin x - \int 2 \sin x dx) \\
= & -x^2 \cos x + 2x \sin x + 2 \cos x + C
\end{aligned}
$$

□

4. 求函數 $y = \sin x$, $y = \cos x$ 的曲線在 $x = 0$ 到 $x = \pi$ 之間所夾的面積大小.

Sol.



先找出交點以分段計算：

$$
\begin{aligned}
&\text{令 } \sin x - \cos x = 0 \;\Rightarrow\; x = \frac{\pi}{4} \\
\Rightarrow\; & \int_0^{\pi} |\sin - \cos x| dx \\
=\; & \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x) dx \\
=\; & \left(\sin x + \cos x \Big|_{x=0}^{x=\frac{\pi}{4}} \right) - \left(\cos x + \sin x \Big|_{x=\frac{\pi}{4}}^{x=\pi} \right) \\
=\; & (\sqrt{2} - 1) - (-1 - (\sqrt{2})) \\
=\; & \sqrt{2} - 1 + 1 + \sqrt{2} \\
=\; & 2\sqrt{2}
\end{aligned}
$$

□

5. 求函數 $f(x)=\dfrac{1}{2}x^2$ 由 $x=0$ 到 $x=1$ 的曲線長度.

Sol.

$$
\begin{aligned}
& f'(x)=x \\
\text{弧長公式：} & \int_0^1 \sqrt{1+(y')^2}dx \\
&= \int_0^1 \sqrt{1+x^2}dx \\
& \left(\text{令} \begin{array}{c} x=\tan\theta \\ dx=\sec^2\theta d\theta \end{array} \left\{\begin{array}{l} x=1 \\ x=0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} \theta=\frac{\pi}{4} \\ \theta=0 \end{array}\right.\right) \\
&= \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2\theta}\sec^2\theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta \\
&= \int_0^{\frac{\pi}{4}} \sec^2\theta\sec\theta d\theta \\
& \left(\text{令} \begin{array}{cc} u=\sec\theta, & du=\sec\theta\tan\theta d\theta \\ dv=\sec^2\theta d\theta, & v=\tan\theta \end{array}\right) \\
&= \sec\theta\tan\theta \Big|_{\theta=0}^{\theta=\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta\tan^2\theta d\theta \\
&= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec\theta(\sec^2\theta-1)d\theta \\
&= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec\theta^3 d\theta + \int_0^{\frac{\pi}{4}} \sec\theta d\theta \\
&= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta + \ln|\sec\theta+\tan\theta| \Big|_{\theta=0}^{\theta=\frac{\pi}{4}} \\
&= \sqrt{2} - \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta + \ln(1+\sqrt{2}) \\
& \left(\text{將 } \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta \text{ 移至等式左側}\right) \\
& \Rightarrow 2\int_0^{\frac{\pi}{4}} \sec^3\theta d\theta = \sqrt{2}+\ln(1+\sqrt{2}) \\
& \Rightarrow \int_0^{\frac{\pi}{4}} \sec^3\theta d\theta = \frac{1}{2}\left(\sqrt{2}+\ln(1+\sqrt{2})\right)
\end{aligned}
$$

□

6. 求 $x = \dfrac{\sqrt{2y}}{1+y^2}$, $y = 1$ 及 $x = 0$ 所圍成的區域繞 y 軸旋轉之體積.

Sol.

注意:

(1) $y = f(x)$ 繞 x 軸 : $V = \int_a^b \pi y^2 dx$	(2) $x = f(y)$ 繞 y 軸 : $V = \int_a^b \pi x^2 dy$
(3) $y = f(x)$ 繞 y 軸 : $V = \int_a^b 2\pi xy dx$	(4) $x = f(y)$ 繞 x 軸 : $V = \int_a^b 2\pi yx dy$

$$\begin{aligned} V &= \int_0^1 \pi x^2 dy \\ &= \int_0^1 \pi \left(\frac{\sqrt{2y}}{1+y^2}\right)^2 dy \\ &= \pi \int_0^1 \frac{2y}{(1+y^2)^2} dy \\ &= \frac{-\pi}{1+y^2}\Bigg|_{y=0}^{y=1} \\ &= \pi\left(-\frac{1}{2}+1\right) \\ &= \frac{\pi}{2} \end{aligned}$$

□

7. 求 $y = \dfrac{\tan x}{x}$, $y = \dfrac{4}{\pi}$ 及 $x = 0$ 所圍成的區域繞 y 軸旋轉之體積.

Sol.

注意:

(1) $y = f(x)$ 繞 x 軸 : $V = \int_a^b \pi y^2 dx$	(2) $x = f(y)$ 繞 y 軸 : $V = \int_a^b \pi x^2 dy$
(3) $y = f(x)$ 繞 y 軸 : $V = \int_a^b 2\pi xy dx$	(4) $x = f(y)$ 繞 x 軸 : $V = \int_a^b 2\pi yx dy$

$$\begin{aligned} V &= 2\pi \int_0^{\frac{\pi}{4}} x\left(\frac{4}{\pi} - \frac{\tan x}{x}\right) dx \\ &= \int_0^{\frac{\pi}{4}} 8x dx - 2\pi \int_0^{\frac{\pi}{4}} \tan x dx \end{aligned}$$

$$
\begin{aligned}
&= \left(4x^2 \Big|_{x=0}^{x=\frac{\pi}{4}}\right) - 2\pi\left(\ln|\sec x| \Big|_{x=0}^{x=\frac{\pi}{4}}\right)\\
&= \frac{\pi^2}{4} - 2\pi(\ln(\sqrt{2}))\\
&= \frac{\pi^2}{4} - \pi\ln 2
\end{aligned}
$$

□

8. 令函數 $f(x) = e^{-x^2}$.

(a) 求 $f(x)$ 對 $x = 0$ 的泰勒展開式, 並寫出一般項.

(b) 以 (a) 中非零的前三項估計積分 $\int_0^{\frac{1}{2}} f(x)dx$, 誤差忽略不計.

Sol.

(a) 已知

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

令 $y = -x^2$

$$
\begin{aligned}
\Rightarrow e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} e^{-x^2} dx\\
&\approx \int_0^{\frac{1}{2}} \left(1 - x^2 + \frac{x^4}{2!}\right) dx\\
&= x - \frac{x^3}{3} + \frac{x^5}{10} \Big|_{x=0}^{x=\frac{1}{2}}\\
&= \frac{1}{2} - \frac{1}{24} + \frac{1}{320}\\
&= \frac{443}{960}
\end{aligned}
$$

□