101-2 微乙期末考試題及詳解

1. 求 $y(t)$ 滿足 $(t^2+1)\,y' = \frac{1}{y}$ 且 $y(0)=2$

Sol.

$$\begin{aligned}
\left(t^2+1\right)\frac{dy}{dt} = \frac{1}{y} &\Rightarrow ydy = \frac{dt}{1+t^2} \\
&\Rightarrow \int ydy = \int \frac{dt}{1+t^2} \\
&\Rightarrow \frac{y^2}{2} = \tan^{-1}(t) + c \\
&\Rightarrow y^2 = 2\tan^{-1}(t) + c
\end{aligned}$$

$y(0)=2$ 代入得

$$4 = 2\tan^{-1}(0) + c \Rightarrow c = 4$$

則

$$y = \sqrt{2\tan^{-1}(t) + 4}$$

(P.S. 不為負 $\rightarrow y(0) = 2 > 0$)

□

2. 求微分方程 $y' - \frac{2}{t}y = t$,$y(1) = -1$ 之解

Sol.

題型:$y' + P(t)\,y = Q(t)$

步驟:

(i) 求積分因子

$$I(t) = e^{\int P(t)dt}$$

(ii) 再求通解

$$I(t)\,y = \int I(t)\,Q(t)\,dt + c$$

(i)

$$I(t) = e^{\int -\frac{2}{t}dt} = e^{-2\ln|t|} = e^{\ln|t^{-2}|} = \frac{1}{t^2}$$

(ii)

$$\frac{1}{t^2}y = \int \frac{1}{t^2}tdt \ \Rightarrow \frac{y}{t^2} = \ln|t| + c$$
$$\Rightarrow y = t^2(\ln|t| + c)$$

$$y(1) = -1$$

代入得 $-1 = c$

$$y = t^2(\ln|t| - 1)$$

□

3. 令 $V_2 = k$, $k \geq 2$, 表示在連續進行白努利試驗時, 到第 k 次試驗才出現第 2 次"+" 事件的隨機變數, 求 $P(V_2 = 10)$.(設白努利試驗出現"+" 機率為 p, 出現"-" 機率為 q, $p + q = 1, 0 < p, q < 1$)

Sol.

出現 2 次 "+" 機率 p^2, 出現 8 次 "-" 機率 q^8,因此

$$P(V_2 = 10) = C_1^9\, p^2 q^8 \quad (\text{第十次必定為 "+" 所以是} C_1^9)$$

□

4. 計算 $\int_{-\infty}^{\infty} xe^{-(x-1)^2} dx$。$\left(已知 \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\right)$

Sol.

(令 $u = x - 1 \Rightarrow du = dx$)

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-(x-1)^2dx} &= \int_{-\infty}^{\infty} (u+1)\, e^{-u^2} du \\
&= \int_{-\infty}^{\infty} ue^{-u^2} du + \int_{-\infty}^{\infty} e^{-u^2} du \\
&= \lim_{a\to-\infty} \int_{a}^{0} ue^{-u^2} du + \lim_{b\to\infty} \int_{0}^{b} ue^{-u^2} du + \sqrt{\pi} \\
&= \lim_{a\to-\infty} \left(-\frac{e^{-u^2}}{2} \Bigg|_{u=a}^{u=0} \right) + \lim_{b\to\infty} \left(-\frac{e^{-u^2}}{2} \Bigg|_{u=0}^{u=b} \right) + \sqrt{\pi} \\
&= -\frac{1}{2} + \frac{1}{2} + \sqrt{\pi} \\
&= \sqrt{\pi}
\end{aligned}$$

□

5.

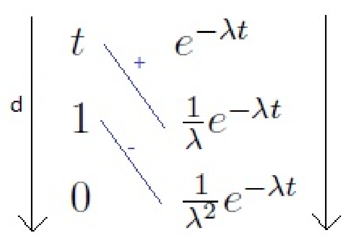
$$f_X(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

求 $E(X)$ 及 $Var(X)$

Sol.

(a)

$$\begin{aligned}
\int_{0}^{\infty} t\lambda e^{-\lambda t} dt &= \lambda \left[\left(-\frac{t}{\lambda} - \frac{1}{\lambda^2} \right) e^{-\lambda t} \Bigg|_{t=0}^{t\to\infty} \right] \\
&= \lambda \left(\frac{1}{\lambda^2} \right) \\
&= \frac{1}{\lambda}
\end{aligned}$$

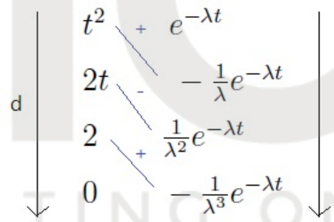


P.S. 補充使用分部積分的過程：

$$
\begin{aligned}
& \int_0^{\infty} t\lambda e^{-\lambda t} dt \\
& \left(令 \begin{matrix} u = t & du = dt \\ dv = \lambda e^{-\lambda t} dt & v = -e^{-\lambda t} \end{matrix} \right) \\
= & -te^{-\lambda t} \Big|_0^{\infty} - \int_0^{\infty} -e^{-\lambda t} dt \\
= & -\frac{e^{-\lambda t}}{\lambda} \Big|_0^{\infty} \\
= & \frac{1}{\lambda}
\end{aligned}
$$

(b)

$$
\begin{aligned}
E\left(x^2\right) & = \int_0^{\infty} t^2 \lambda e^{-\lambda t} dt \\
& = \lambda \left[e^{-\lambda t} \left(-\frac{t^2}{\lambda} - \frac{2t}{\lambda^2} - \frac{2}{\lambda^3} \right) \Big|_{t=0}^{t \to \infty} \right] \\
& = \lambda \left(\frac{2}{\lambda^3} \right) \\
& = \frac{2}{\lambda^2}
\end{aligned}
$$



$$
\begin{aligned}
V(X) & = E\left(X^2\right) - (E(X))^2 \\
& = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda} \right)^2 \\
& = \frac{1}{\lambda^2}
\end{aligned}
$$

P.S. 補充使用分部積分的過程：

$$
\begin{aligned}
& \int_0^{\infty} t^2 \lambda e^{-\lambda t} dt \\
& \left(令 \begin{matrix} u = t^2 & du = 2tdt \\ dv = \lambda e^{-\lambda t} dt & v = -e^{-\lambda t} \end{matrix} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= -t^2e^{-\lambda t}\Big|_0^\infty - \int_0^\infty -e^{-\lambda t}2tdt \\
&= \frac{2}{\lambda}\int_0^\infty t\lambda e^{-\lambda t}dt \quad \text{(套用 (a) 的結果)} \\
&= \frac{2}{\lambda^2}
\end{aligned}
$$

□

6. 根據統計, 在 15 歲到 35 歲之間酒駕死亡的機率密度為 $f(x) = \frac{c}{x^2}$, $15 \le x \le 35$

   (a) 求 c 之值

   (b) 求年齡在 21 到 25 歲間酒駕死亡的機率為何?

Sol.

(a)

$$
\int_{15}^{35} \frac{c}{x^2}dx = -\frac{c}{x}\Big|_{x=15}^{x=35} = \frac{c}{15} - \frac{c}{35} = 1
$$

$$
\begin{aligned}
\frac{105}{15}c - \frac{105}{35}c &= 105 \\
7c - 3c &= 105 \\
4c &= 105 \\
c &= \frac{105}{4}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\int_{21}^{25} \frac{1}{x^2}\frac{105}{4}dx &= \frac{105}{4}\left(-\frac{1}{x}\Big|_{x=21}^{x=25}\right) \\
&= \frac{105}{4}\left(\frac{1}{21} - \frac{1}{25}\right) \\
&= \frac{5}{4} - \frac{21}{20} \\
&= \frac{4}{20} \\
&= \frac{1}{5}
\end{aligned}
$$

□

7. 某十字路口每年平均發生 36 次交通事故 (註: 一年以 360 天計), 假設事故之發生遵守 Poisson 分配

   (a) 求在 15 天之內恰有一件交通事故的機率

   (b) 求在 15 天之內有兩件或兩件以上交通事故的機率

Sol.

λ: 平均時間內發生次數
平均每天發生 $\lambda = \frac{36}{360} = \frac{1}{10}$ 件事故
15 天平均發生 $0.1 \times 15 = 1.5$ 件事故

(a)

$$\text{Poisson}\,(\lambda = 1.5) = \frac{1.5^x}{x!}e^{-1.5}$$

$$P\,(X = 1) = \frac{1.5^1}{1!}e^{-1.5} = 1.5e^{-1.5}$$

(b)

$$\begin{aligned} &1 - P\,(X = 0) - P\,(X = 1) \\ &= 1 - \frac{1.5^0}{0!}e^{-1.5} - \frac{1.5^1}{1!}e^{-1.5} \\ &= 1 - e^{-1.5} - 1.5e^{-1.5} \\ &= 1 - 2.5e^{-1.5} \end{aligned}$$

□

8. 設 X,Y 為隨機變數取值 1 或 2. 已知

$$P(X=1,Y=1)=\frac{1}{14},\quad P(X=1,Y=2)=\frac{5}{14},$$
$$P(X=2,Y=1)=\frac{6}{14},\quad P(X=2,Y=2)=\frac{2}{14}$$

求 $P(X=1)$,$P(X=2)$ 及 $E(X)$

Sol.

(a)

$$\begin{aligned} P(X=1) &= \sum_Y P_{X,Y}(X=1,Y) \\ &= \frac{1}{14}+\frac{5}{14} \\ &= \frac{3}{7} \end{aligned}$$

(b)

$$\begin{aligned} P(X=2) &= \sum_Y P_{X,Y}(X=2,Y) \\ &= \frac{6}{14}+\frac{2}{14} \\ &= \frac{4}{7} \end{aligned}$$

(c)

$$\begin{aligned} E(X) &= \sum_X X f_x(X) \\ &= 1\times\frac{3}{7}+2\times\frac{4}{7} \\ &= \frac{11}{7} \end{aligned}$$

□