# 101-2 微乙期中考試題及詳解

1. 求過曲面 $z=e^{x^2y-1}$ 上點 $(1,1,1)$ 的切平面方程式

*Sol.*

令

$$\phi(x,y,z)=e^{x^2y-1}-z=0$$

$$\begin{aligned}\nabla\phi(x,y,z)&=\frac{\partial\phi}{\partial x}\vec{i}+\frac{\partial\phi}{\partial y}\vec{j}+\frac{\partial\phi}{\partial z}\vec{k}\\&=2xye^{x^2y-1}\vec{i}+x^2e^{x^2y-1}\vec{j}-\vec{k}\end{aligned}$$

$$\nabla\phi(1,1,1)=2\vec{i}+\vec{j}-\vec{k}$$

切平面方程式:

$$2(x-1)+(y-1)-(z-1)=0$$

$$\Rightarrow 2x+y-z=2$$

□

2. 求 $f(x,y) = y^4 + 2xy^3 + x^2y^2$

   (a) 在點 $(0,1)$ 的梯度

   (b) 在點 $(0,1)$ 沿著 $(1,2)$ 方向的方向導數

*Sol.*

(a)

$$\begin{aligned}\nabla f(x,y) &= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \\ &= \left(2y^3 + 2xy^2\right)\vec{i} + \left(4y^3 + 6xy^2 + 2x^2y\right)\vec{j}\end{aligned}$$

$$\nabla f(0,1) = 2\vec{i} + 4\vec{j}$$

(b)

$$\begin{aligned}&\nabla f(0,1) \cdot \frac{\vec{u}}{\|\vec{u}\|} \quad \left(\|u\| = \sqrt{1^2+2^2} = \sqrt{5}\right) \\ &= (2,4) \cdot \frac{(1,2)}{\sqrt{5}} \\ &= \frac{10}{\sqrt{5}} = 2\sqrt{5}\end{aligned}$$

□

3. 求函數 $f(x,y) = xy - x^2y - xy^2$ 的

   (a) 極值候選點

   (b) 並討論其極值性質 (含鞍點)

*Sol.*

(a) 令

$$\begin{cases} f_x = y - 2xy - y^2 = 0 \\ f_y = x - x^2 - 2xy = 0 \end{cases}$$

$$\Rightarrow \begin{cases} y(1-2x-y) = 0 \\ x(1-x-2y) = 0 \end{cases}$$

分情況討論極值候選點的解：

(i) $x=0$ 且 $y=0$

$$(x,y)=(0,0)$$

(ii) $x\neq 0$ 且 $y\neq 0$，則

$$\begin{cases} 2x+y=1 \\ x+2y=1 \end{cases}$$

$$\Rightarrow\ x=y=\frac{1}{3}$$

$$\Rightarrow\ (x,y)=(\frac{1}{3},\frac{1}{3})$$

(iii) $y=0$ 但 $x\neq 0$，則

$$x(1-x+0)=0$$

$$\Rightarrow\ x=1$$

$$\Rightarrow\ (x,y)=(1,0)$$

(iv) $x=0$ 但 $y\neq 0$，則

$$y(1-0-y)=0$$

$$\Rightarrow\ y=1$$

$$\Rightarrow\ (x,y)=(0,1)$$

(b)

$$D(x,y)=\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}=\begin{vmatrix} -2y & 1-2x-2y \\ 1-2x-2y & -2x \end{vmatrix}$$

$$\begin{cases} D\left(x,y\right)<0 \Rightarrow \text{saddle point} \\ D\left(x,y\right)>0, f_{xx}>0 \rightarrow \min \\ D\left(x,y\right)>0, f_{xx}<0 \rightarrow \max \end{cases}$$

(i) $D(0,0)=\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}=-1<0 \Rightarrow$ saddle point

(ii) $D\left(1,0\right)=\begin{vmatrix} 0 & -1 \\ -1 & -2 \end{vmatrix}=-1<0 \Rightarrow$ saddle point

(iii) $D\left(0,1\right)=\begin{vmatrix} -2 & -1 \\ -1 & 0 \end{vmatrix}=-1<0 \Rightarrow$ saddle point

(iv) $D\left(\frac{1}{3},\frac{1}{3}\right)=\begin{vmatrix} -\frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{2}{3} \end{vmatrix}=\frac{4}{9}-\frac{1}{9}=\frac{1}{3}>0$, 且 $f_{xx}=-\frac{2}{3}<0 \Rightarrow$ 極大值

$\square$

4. 使用 Lagrange 乘子法求從點 $(0,0)$ 到曲線 $y=x^2-\frac{5}{4}$ 的最短距離

*Sol.*

距離：$\sqrt{x^2+y^2}$

$$\begin{cases} \min\ (x,y)=x^2+y^2 \\ y=x^2-\dfrac{5}{4} \end{cases}$$

$L(x,y,\lambda)=x^2+y^2+\lambda\left(y-x^2+\frac{5}{4}\right)$

令

$$\begin{cases} \dfrac{\partial L}{\partial x}=2x-2x\lambda=0 & ① \\ \dfrac{\partial L}{\partial y}=2y+\lambda=0 & ② \\ \dfrac{\partial L}{\partial \lambda}=y-x^2+\dfrac{5}{4}=0 & ③ \end{cases}$$

$$①\,②\text{ 整理得 } \Rightarrow \begin{cases} 2x(1-\lambda)=0 \\ 2y+\lambda=0 \end{cases}$$

$$\text{將 ① 分情況討論：} \Rightarrow \begin{cases} \text{當 } \lambda=1,\text{ 帶入 ② 得 } y=-\dfrac{1}{2},\text{ 再帶入 ③ 得} x=\pm\dfrac{\sqrt{3}}{2} \\ \text{當 } x=0,\text{ 帶入 ③ 得 } y=-\dfrac{5}{4} \end{cases}$$

$$\begin{cases} f\left(\pm\dfrac{\sqrt{3}}{2},-\dfrac{1}{2}\right)=\dfrac{3}{4}+\dfrac{1}{4}=1\ \rightarrow \min \\ f\left(0,-\dfrac{5}{4}\right)=\dfrac{25}{16} \end{cases}$$

□

5. 求 $\iint_\Omega \frac{1}{(1+x+y)^2}dA$, 其中 $\Omega=[0,2]\times[0,3]$

*Sol.*

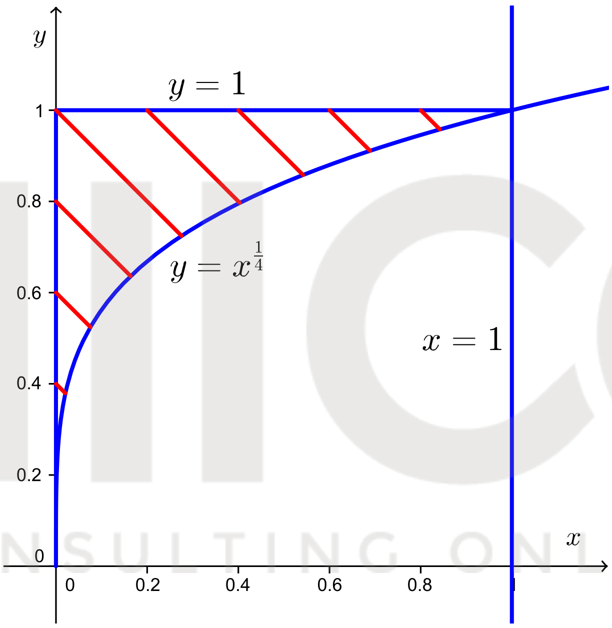
$$\begin{aligned} &\int_0^3\int_0^2 \frac{1}{(1+x+y)^2}dxdy \\ =&\int_0^3\left(\frac{-1}{1+x+y}\Bigg|_{x=0}^{x=2}\right)dy \end{aligned}$$

$$
\begin{aligned}
&= \int_0^3 \left( \frac{1}{1+y} - \frac{1}{3+y} \right) dy \\
&= \ln(1+y) - \ln(3+y) \Big|_{y=0}^{y=3} \\
&= \ln 4 - \ln 6 + \ln 3 \\
&= \ln \left( \frac{4 \times 3}{6} \right) \\
&= \ln 2
\end{aligned}
$$

□

6. 求 $\int_0^1 \int_{x^{\frac{1}{4}}}^1 \frac{1}{1+y^5} dy dx$ 之值

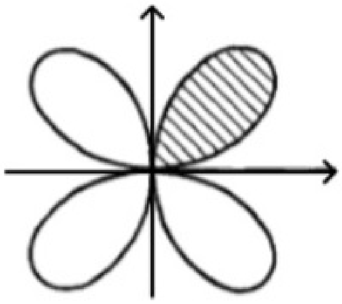
*Sol.*




$$
\begin{aligned}
&\int_0^1 \int_{x^{\frac{1}{4}}}^1 \frac{1}{1+y^5} dy dx \\
&= \int_0^1 \int_0^{y^4} \frac{1}{1+y^5} dx dy \\
&= \int_0^1 \frac{y^4}{1+y^5} dy \\
&= \frac{1}{5} \ln\left(1+y^5\right) \Big|_{y=0}^{y=1} \\
&= \frac{1}{5} \ln 2
\end{aligned}
$$

$\square$

7. $r = \sin 2\theta$ 之圖如下, 求 $\iint_{\Omega} xy\, dA$, 其中 $\Omega$ 為第一象限之一葉 (提示: $\cos\theta \sin\theta = \frac{\sin 2\theta}{2}$)



*Sol.*

令

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\mid J \mid = r$$

$$\begin{aligned}
&\int\int_{\Omega} xy\, dA \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r\cos\theta r\sin\theta r dr d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} \frac{1}{2} r^3 \sin 2\theta dr d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{4}\sin^5 2\theta d\theta \\
&= \frac{1}{8}\int_0^{\frac{\pi}{2}} -\frac{1}{2}\left(1-\cos^2 2\theta\right)^2 d\cos 2\theta
\end{aligned}$$

令 $u = \cos 2\theta \Rightarrow du = d\cos 2\theta$

$$\begin{cases} \theta = \frac{\pi}{2} \to u = -1 \\ \theta = 0 \to u = 1 \end{cases}$$

$$\begin{aligned}
&\Rightarrow \frac{1}{16}\int_{-1}^{1} \left(1-u^2\right)^2 du \\
&= \frac{1}{16}\int_{-1}^{1} 1 - 2u^2 + u^4 du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{16}\left(u - \frac{2u^3}{3} + \frac{u^5}{5}\ \Big|_{u=-1}^{u=1}\right) \\
&= \frac{1}{16}\left(\frac{8}{15} + \frac{8}{15}\right) \\
&= \frac{1}{16} \cdot \frac{16}{15} \\
&= \frac{1}{15}
\end{aligned}
$$

□

8. 求 $\iint_\Omega (3x+y)^6\, dA$, 其中 $\Omega$ 為 $x+y=\pm 1$ 及 $3x+y=\pm 1$ 所包圍的平行四邊形

*Sol.*

令

$$
\begin{cases} u = x+y \\ v = 3x+y \end{cases}
\Rightarrow \begin{cases} v-u = 2x \\ 3u-v = 2y \end{cases}
\Rightarrow \begin{cases} x = \dfrac{v-u}{2} \\ y = \dfrac{3u-v}{2} \end{cases}
$$

$$
|\, J\, | = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} \end{vmatrix} = \left| \frac{1}{4} - \frac{3}{4} \right| = \frac{1}{2}
$$

$$
\begin{aligned}
&\iint_\Omega (3x+y)^6\, dA \\
&= \int_{-1}^{1}\int_{-1}^{1} \frac{1}{2} v^6 du dv \\
&= \int_{-1}^{1} v^6 dv \\
&= \frac{2}{7}
\end{aligned}
$$

□