102-1 微甲 07-11 班期末考試題及詳解

1. Let $h(x) = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x}), 0 \le x \le 1$.

   (a) Find the length of the curve $y = h(x)$.

   (b) Find the area of the surface generated by rotating the curve $y = h(x)$ about the x-axis.

Sol.

(a)

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{1}{2}(1-2x)}{\sqrt{x - x^2}} + \frac{1}{\sqrt{1 - (\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} \\
&= \sqrt{\frac{1-x}{x}}, \quad 0 < x < 1 \\
\Rightarrow ds &= \sqrt{\frac{1}{x}} dx \\
s &= \int_0^1 \frac{1}{\sqrt{x}} dx \\
&= 2\sqrt{x}\Big|_0^1 \\
&= 2
\end{aligned}$$

(b)

$$\begin{aligned}
A &= \int 2\pi y ds \\
&= 2\pi \int_0^1 \left(\sqrt{x - x^2} + \sin^{-1}(\sqrt{x})\right) \frac{1}{\sqrt{x}} dx \\
&= 2\pi \int_0^1 \left(\sqrt{1-x} + \frac{\sin^{-1}(\sqrt{x})}{\sqrt{x}}\right) dx
\end{aligned}$$

$$= 2\pi\left[\frac{-2}{3}(1-x)^{\frac{3}{2}}\Big|_0^1 + 2\sqrt{x}\sin^{-1}(\sqrt{x})\Big|_0^1 - \int_0^1 \sqrt{x}\frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}}dx\right]$$
$$= 2\pi(\pi - \frac{4}{3})$$

□

2. Solve $xy' - 3y = 5x^3$

   (a) with the initial condition $y(1) = 2$.

   (b) with the initial condition $y(-1) = 2$.

Sol.

$$xy' - 3y = 5x^3 \quad \Rightarrow \; y' - \frac{3}{x}y = 5x^2$$

$$(Iy)' = I(y' - \frac{3}{x}y) = 5x^2 I$$
$$I'y + Iy' = Iy' - \frac{3}{x}Iy$$
$$\Rightarrow \frac{I'}{I} = -\frac{3}{x}$$
$$\Rightarrow \ln I = -3\ln|x| + C, \text{ let } C = 0$$
$$\Rightarrow I = \frac{1}{x^3}$$
$$\therefore (\frac{1}{x^3}y)' = \frac{5}{x}$$
$$\Rightarrow \frac{1}{x^3}y = 5\ln|x| + C$$
$$\Rightarrow y = 5x^3\ln|x| + Cx^3$$

(a) $y(1) = 2, \; C = 2,$

$$\Rightarrow y = 5x^3\ln|x| + 2x^3$$

(b) $y(-1) = 2, \; C = -2,$

$$\Rightarrow y = 5x^3\ln|x| - 2x^3$$

□

3. Let $y = h(x)$ be decreasing on $[0, \frac{\pi}{2})$ and is continuously differentiable on $(0, \frac{\pi}{2})$ with $h(0) = 0$. Let $s(x)$ denote the arc length of $y = h(x)$ from $(0, 0)$ to $(x, h(x))$.

(a) Write down the formula for $s(x)$.

(b) Suppose that $s(x)$ is also given by $s(x) = \int_0^x e^{-h(t)} dt$. Find the function $h(x)$ explicitly.

(c) Find the function $s(x)$ explicitly.

Sol.

(a)

$$s(x) = \int_0^x \sqrt{1 + (h'(t))^2} dt$$

$$\left(ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \right)$$

(b)

$$\int_0^x \sqrt{1 + (h'(t))^2} dt = \int_0^x e^{-h(t)} dt$$

$$\Rightarrow \sqrt{1 + (h'(x))^2} = e^{-h(x)}$$

$$\Rightarrow 1 + (h'(x))^2 = e^{-2h(x)}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = e^{-2y}$$

$$\Rightarrow \frac{dy}{dx} = -\sqrt{e^{-2y} - 1}$$

$$\Rightarrow -\frac{1}{\sqrt{e^{-2y} - 1}} dy = dx$$

Let $e^{-y} = \sec\theta$, then

$$-e^{-y} dy = \sec\theta \tan\theta d\theta$$

$$-dy = e^{y} \sec\theta \tan\theta d\theta$$

$$= \cos\theta \sec\theta \tan\theta d\theta$$

Therefore,

$$\int \frac{\cos\theta \sec\theta \tan\theta}{\tan\theta} d\theta = \int dx$$

$$\Rightarrow \theta + C = x$$

$$\Rightarrow x = \sec^{-1}(e^{-y}) + C$$

$$\because (x, y) = (0, 0) \quad \Rightarrow \quad C = 0$$

$$\Rightarrow e^{-y} = \sec x$$

$$\Rightarrow h(x) = y = -\ln|\sec x| = \ln|\cos x|$$

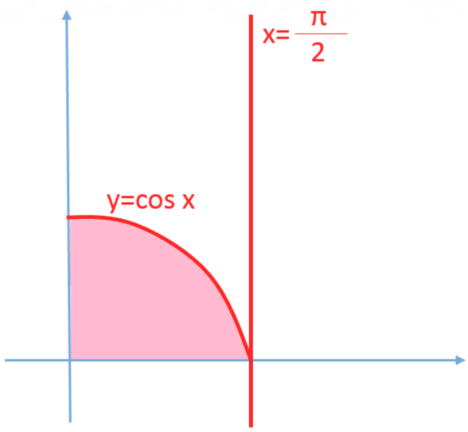
(c)

$$\begin{aligned} s(x) &= \int_0^x \sqrt{1 + \left(\frac{\sin t}{\cos t}\right)^2} dt \\ &= \int_0^x \sec t dt \\ &= \ln|\sec x + \tan x| \end{aligned}$$

□

4. Let Ω be the region bounded by $y = \cos x, y = 0, x = 0$ and $x = \frac{\pi}{2}$.
   (a) Find the volume of the solid obtained by revolving Ω about x-axis.
   (b) Find the volume of the solid obtained by revolving Ω about y-axis.
   (c) Find the centroid of Ω.

Sol.



(a)

$$\begin{aligned} V &= \int_0^{\pi/2} \pi y^2 dx = \pi \int_0^{\pi/2} \cos^2(x) dx \\ &= \pi \int_0^{\pi/2} \frac{\cos 2x + 1}{2} dx \end{aligned}$$

$$
\begin{aligned}
&= \pi \left(\frac{\sin 2x}{4} + \frac{x}{2} \right) \Big|_0^{\pi/2} \\
&= \frac{\pi^2}{4}
\end{aligned}
$$

(b)

$$
\begin{aligned}
V &= \int \pi x^2 dy = \int \pi x^2 (-\sin x) dx \\
&= -\pi \int \underbrace{x^2}_{A} \underbrace{\sin x dx}_{dB} \\
&= -\pi \left[x^2(-\cos x)\Big|_{\frac{\pi}{2}}^{0} - \int (-\cos x) 2x dx \right] \\
&= -\pi \left(2 \int \underbrace{x}_{A} \underbrace{\cos x dx}_{dB} \right) \\
&= -2\pi \left(x \sin x \Big|_{\frac{\pi}{2}}^{0} - \int_{\frac{\pi}{2}}^{0} \sin x dx \right) \\
&= \pi^2 - 2\pi
\end{aligned}
$$

(c)

$$
\begin{aligned}
\bar{x} &= \frac{\int_0^{\pi/2} x \cos x dx}{\int_0^{\pi/2} \cos x dx} = \frac{\pi}{2} - 1 \\
\bar{y} &= \frac{\int_0^{\pi/2} \frac{1}{2} \cos^2 x dx}{\int_0^{\pi/2} \cos x dx} = \frac{\pi}{8}
\end{aligned}
$$

or use Pappus Theorem:

$$2\pi A\bar{x} = V (\text{resolve about } y\text{-axis})$$

$$2\pi A\bar{y} = V (\text{resolve about } x\text{-axis})$$

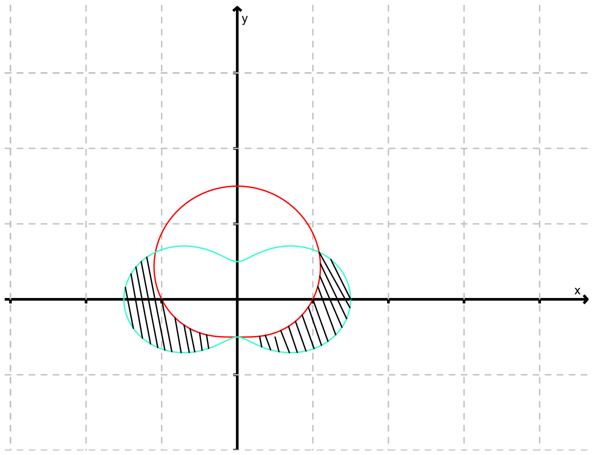
□

5. Find the area of the region that lies inside the curve $r = 2 + \cos 2\theta$ but outside the curve $r = 2 + \sin\theta$.

Sol.

(i) Find intersections

$$2 + \sin\theta = 2 + \cos 2\theta$$



$$\sin\theta = 1 - 2\sin^2\theta$$
$$\Rightarrow \ \sin\theta = -1 \text{ or } \frac{1}{2}$$
$$\Rightarrow \theta = -\frac{\pi}{2}, \ \frac{\pi}{6}, \ \frac{5\pi}{6}$$

(ii)

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} [(2+\cos 2\theta)^2 - (2+\sin\theta)^2]d\theta \times 2 \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (4\cos 2\theta + \cos^2 2\theta - 4\sin\theta - \sin^2\theta)d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (4\cos 2\theta + \frac{1+\cos 4\theta}{2} - 4\sin\theta - \frac{1-\cos 2\theta}{2})d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\frac{9}{2}\cos 2\theta + \frac{1}{2}\cos 4\theta - 4\sin\theta\right) d\theta \\
&= \frac{9}{4}\sin 2\theta + \frac{1}{8}\sin 4\theta + 4\cos\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \\
&= \frac{51}{16}\sqrt{3}
\end{aligned}$$

□

6. Find the arc length of the curve $x = t\sin 2t, y = t\cos 2t, 0 \le t \le 1$.

Sol.

$$dl = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$
$$x'(t) = (t\sin 2t)' = \sin(2t) + 2t\cos(2t)$$

$$y'(t) = (t\cos 2t)' = \cos(2t) - 2t\sin(2t)$$

$$L = \int_0^1 \sqrt{(x')^2 + (y')^2}dt = \int_0^1 \sqrt{1+4t^2}dt$$

Let $2t = \tan\theta \;\Rightarrow dt = \frac{1}{2}\sec^2\theta d\theta$

$$\begin{aligned} L &= \frac{1}{2}\int_0^{\tan^{-1}(2)} |\sec\theta|\sec^2\theta d\theta \\ &= \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3\theta d\theta \\ &= \frac{1}{2}\cdot\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)\Big|_0^{\tan^{-1}(2)} \\ &= \frac{1}{4}[2\sqrt{5} + \ln(2+\sqrt{5})] \end{aligned}$$

Note that

$$\begin{aligned} \int \sec^3\theta d\theta &= \int \sec\theta\sec^2\theta d\theta \\ &= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta \\ &= \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| - \int\sec^3\theta d\theta \end{aligned}$$

$$\Rightarrow \int \sec^3\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)$$

□

7. Let $P(x) = x^3 + x^2 + x + 1$ and $Q(x) = x^3 - x^2 + x + 1$. Evaluate $\int \frac{Q(x)}{P(x)}dx$. Note that $P(-1) = 0$.

Sol.

$$\begin{aligned} &\int \frac{x^3 - x^2 + x + 1}{x^3 + x^2 + x + 1}dx \\ =& \int 1 - \frac{2x^2}{x^3+x^2+x+1}dx \\ =& \int 1 - \frac{2x^2}{(x^2+1)(x+1)}dx \\ =& \int 1 - \frac{1}{x+1} + \frac{1-x}{x^2+1}dx \\ =& \, x - \ln|x+1| + \tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C \end{aligned}$$

□

8. Determine the values of $\alpha > 0$ such that $\displaystyle\int_1^\infty \frac{\ln x}{x^\alpha}dx$ is convergent.

Sol.

(a) If $\alpha \neq 1$

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^\alpha}dx &= \ln x \frac{x^{1-\alpha}}{1-\alpha}\bigg|_1^\infty - \frac{1}{1-\alpha}\int_1^\infty x^{-\alpha}dx \\ &= \left[\frac{x^{1-\alpha}}{1-\alpha}\ln x - \frac{x^{1-\alpha}}{(1-\alpha)^2}\right]\bigg|_{x\to\infty} + \frac{1}{(1-\alpha)^2}\end{aligned}$$

(i) If $0 < \alpha < 1$

$$\lim_{x\to\infty} \frac{(1-\alpha)\ln x - 1}{(1-\alpha)^2 x^{\alpha-1}} = \infty$$

(ii) If $\alpha > 1$, by L'Hospital Rule,

$$\begin{aligned}&\lim_{x\to\infty} \frac{(1-\alpha)\ln x - 1}{(1-\alpha)^2 x^{\alpha-1}} \\ &= \frac{\frac{1}{x}(1-\alpha)}{(1-\alpha)^2(\alpha-1)x^{\alpha-2}}\bigg|_{x\to\infty} \\ &= \frac{-1}{(1-\alpha)^2 x^{\alpha-1}}\bigg|_{x\to\infty} \\ &= 0\end{aligned}$$

(b) If $\alpha = 1$

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^\alpha}dx &= \int_1^\infty \frac{\ln x}{x}dx \quad (\text{Let } u = \ln x,\ du = \frac{1}{x}dx) \\ &= \frac{1}{2}(\ln x)^2\bigg|_{x\to\infty} \\ &= \infty\end{aligned}$$

$$\therefore \int_1^\infty \frac{\ln x}{x^\alpha}dx \text{ is convergent for } \alpha > 1$$

□