102-1 微甲 07-11 班期中考試題及詳解

1. Find $\frac{d}{dx}(\sec x)^x, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$.

Sol.

$$\begin{aligned}
&\frac{d}{dx}(\sec x)^x \\
&= \frac{d}{dx}\left(e^{x\ln(\sec x)}\right) \\
&= e^{x\ln(\sec x)}\left(x\ln(\sec x)\right)' \\
&= e^{x\ln(\sec x)}\left(\ln(\sec x) + x\cdot\frac{1}{\sec x}\cdot\sec x\tan x\right) \\
&= (\sec x)^x(\ln(\sec x) + x\tan x)
\end{aligned}$$

□

2. Evaluate $\int_0^1 \frac{(2+x)^2}{1+x^2}dx$.

Sol.

$$\begin{aligned}
&\int_0^1 \frac{(2+x)^2}{1+x^2}dx \\
&= \int_0^1 \frac{x^2+4x+4}{1+x^2}dx \\
&= \int_0^1 1 + \frac{4x}{1+x^2} + \frac{3}{1+x^2}dx \\
&= \int_0^1 1dx + 4\int_0^1 \frac{x}{1+x^2}dx + 3\int_0^1 \frac{1}{1+x^2}dx \\
&= x\Big|_0^1 + 4\ln(1+x^2)\cdot\frac{1}{2}\Big|_0^1 + 3\tan^{-1}x\Big|_0^1 \\
&= 1 + 2\ln 2 + \frac{3}{4}\pi
\end{aligned}$$

□

3. (a) Find $\lim_{t\to 0^+} \dfrac{t-\ln(1+t)}{t^2}$

(b) Use (a) to find $\lim_{t\to 0^+} \dfrac{\sqrt{t-\ln(1+t)}}{t}$

Sol.

(a)

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{t-\ln(1+t)}{t^2} \left(\frac{0}{0}, \text{ by L'Hôpital's Rule}\right)\\
=&\lim_{t\to 0^+} \frac{1-\frac{1}{1+t}}{2t}\\
=&\left.\frac{t}{2t(1+t)}\right|_{t\to 0^+}\\
=&\frac{1}{2}
\end{aligned}$$

(b)

$$\begin{aligned}
f(t) &= \frac{t-\ln(1+t)}{t^2}\\
g(t) &= \sqrt{t}
\end{aligned}$$

Since $g(t)$ is continuous at $\frac{1}{2}$

$$\therefore \lim_{t\to 0^+} g(f(t)) = g\left(\lim_{t\to 0^+} f(t)\right) = \sqrt{\frac{1}{2}}$$

□

4. Evaluate $\lim_{x\to\infty} (2^x+3^x+5^x)^{\frac{1}{x}}$

Sol.

- Method 1

$$\begin{aligned}
&\lim_{x\to\infty} (2^x+3^x+5^x)^{\frac{1}{x}}\\
=&\lim_{x\to\infty} e^{\frac{1}{x}\ln(2^x+3^x+5^x)}\\
=&\exp\left(\lim_{x\to\infty} \frac{\ln(2^x+3^x+5^x)}{x}\right) \quad \left(\frac{\infty}{\infty}, \text{ by L'Hôpital's Rule}\right)\\
=&\exp\left(\lim_{x\to\infty} \frac{2^x\ln 2+3^x\ln 3+5^x\ln 5}{2^x+3^x+5^x}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\lim_{x\to\infty} \frac{5^x\left(\ln 5 + \left(\frac{2}{5}\right)^x \ln 2 + \left(\frac{3}{5}\right)^x \ln 3\right)}{5^x(1+(\frac{2}{5})^x+(\frac{3}{5})^x)}\right) \\
&= 5
\end{aligned}
$$

- Method 2

$$
\begin{aligned}
5^x < \quad & 2^x + 3^x + 5^x \quad < 3\cdot 5^x \\
\lim_{x\to\infty}(5^x)^{\frac{1}{x}} \le & \lim_{x\to\infty}(2^x+3^x+5^x)^{\frac{1}{x}} \le \lim_{x\to\infty}(3\cdot 5^x)^{\frac{1}{x}} \\
5 \le & \lim_{x\to\infty}(2^x+3^x+5^x)^{\frac{1}{x}} \le 3^0 \cdot 5 = 5
\end{aligned}
$$

$$
\lim_{x\to\infty}(2^x+3^x+5^x)^{\frac{1}{x}} = 5 \quad \text{(by Squeeze Theorem)}
$$

□

5. Find the linear approximation of the function

$$
g(x) = \sin^{-1}\left(\frac{x-1}{x+1}\right) - \tan^{-1}(\sqrt{x}) \quad \text{at the point } x = 3
$$

Sol.

$$
\begin{aligned}
f(x+\Delta x) &\approx f(x) + \Delta x \cdot f'(x) \\
g(x) &\approx g(3) + g'(3)(x-3) \\
g'(x) &= \frac{\left(\frac{x-1}{x+1}\right)'}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} - \frac{1}{1+(\sqrt{x})^2}(\sqrt{x})' \\
&= \frac{\frac{2}{(x+1)^2}}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} - \frac{1}{1+x}\cdot\frac{1}{2}\frac{1}{\sqrt{x}} \\
g'(3) &= \frac{\frac{1}{8}}{\frac{\sqrt{3}}{2}} - \frac{1}{8\sqrt{3}} = \frac{1}{8\sqrt{3}} \\
g(3) &= \sin^{-1}\left(\frac{1}{2}\right) - \tan^{-1}(\sqrt{3}) = \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6} \\
\therefore g(x) &\approx g(3) + g'(3)(x-3) \\
&= -\frac{\pi}{6} + \frac{1}{8\sqrt{3}}(x-3)
\end{aligned}
$$

□

6. Let $y = f(x)$ satisfy $x^3 + 2xy + y^3 = 13$. Find y' and y'' at the point $x = 1,\ y = 2$.

Sol.

Differentiate with respect to x

$$\begin{aligned} & x^3 + 2xy + y^3 = 13 \\ \Rightarrow\ & 3x^2 + 2y + 2xy' + 3y^2 y' = 0 \\ & y' = -\frac{3x^2 + 2y}{2x + 3y^2} \end{aligned}$$

Differentiate with respect to x again :

$$\begin{aligned} & 6x + 2y' + 2y' + 2xy'' + 6y(y')^2 + 3y^2 y'' = 0 \\ & y'' = -\frac{6x + 4y' + 6y(y')^2}{2x + 3y^2} \end{aligned}$$

at $(1, 2)$:

$$\begin{aligned} y' &= -\frac{3 \cdot 1^2 + 2 \cdot 2}{2 \cdot 1 + 3 \cdot 2^2} = -\frac{7}{14} = -\frac{1}{2} \\ y'' &= -\frac{6 \cdot 1 + 4 \cdot (-\frac{1}{2}) + 6 \cdot 2 \cdot (-\frac{1}{2})^2}{2 \cdot 1 + 3 \cdot 2^2} = -\frac{7}{14} = -\frac{1}{2} \end{aligned}$$

□

7. Evaluate $\lim_{x \to 0^+} \dfrac{\int_0^{x^2} e^t \sqrt{t} \sin \sqrt{t} dt + x \cos x - x}{x^3}$

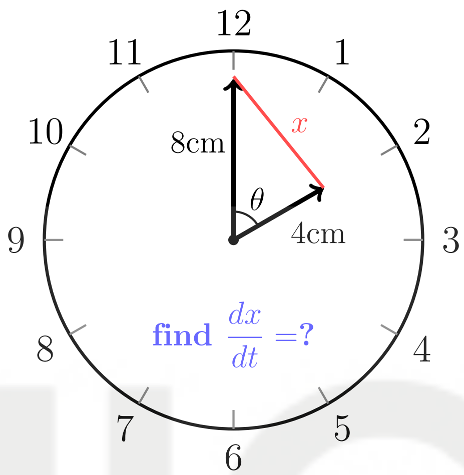
Sol.

$$\begin{aligned} & \lim_{x \to 0^+} \frac{\int_0^{x^2} e^t \sqrt{t} \sin \sqrt{t} dt + x \cos x - x}{x^3} \quad \left(\frac{0}{0}, \text{ by L'Hôpital's Rule}\right) \\ =\ & \lim_{x \to 0^+} \frac{e^{x^2} \sqrt{x^2} \sin \sqrt{x^2} \cdot (x^2)' + \cos x - x \sin x - 1}{3x^2} \\ =\ & \lim_{x \to 0^+} \frac{2e^{x^2} \sin x \cdot x^2 + \cos x - x \sin x - 1}{3x^2} \\ =\ & \lim_{x \to 0^+} \frac{2e^{x^2} \sin x \cdot x^2}{3x^2} + \lim_{x \to 0^+} \frac{\cos x - 1}{3x^2} + \lim_{x \to 0^+} \frac{-\sin x}{3x} \\ =\ & \lim_{x \to 0^+} \frac{2e^{x^2} \sin x}{3} + \lim_{x \to 0^+} \frac{-\cos x}{6} - \frac{1}{3} \lim_{x \to 0^+} \frac{\sin x}{x} \quad \text{(by L'Hôpital twice)} \\ =\ & 0 + \left(-\frac{1}{6}\right) + \left(-\frac{1}{3}\right) = -\frac{1}{2} \end{aligned}$$

□

8. The minute hand (分針) on a clock is 8cm long and the hour hand (時針) is 4cm long. How fast is the distance between the tips of the hands changing at two o'clock? Give your answer in the unit cm/hour.

Sol.



$$x^2 = 8^2 + 4^2 - 2 \cdot 8 \cdot 4 \cos\theta$$
$$2x\frac{dx}{dt} = 64 \sin\theta \frac{d\theta}{dt}$$
$$\theta = 2\pi \cdot \frac{2}{12} = \frac{\pi}{3}$$
$$\sin\theta = \frac{\sqrt{3}}{2}$$
$$x = \sqrt{64 + 16 - 32} = 4\sqrt{3}$$

$$\frac{d\theta}{dt} = \frac{2\pi}{12} - \frac{2\pi}{1} = -\frac{11}{6}\pi \quad \text{(rad/hr)}$$
$$\frac{dx}{dt} = \frac{1}{8\sqrt{3}} \cdot 64 \cdot \frac{\sqrt{3}}{2} \cdot -\frac{11}{6}\pi$$
$$= 4 \cdot -\frac{11}{6}\pi$$
$$= -\frac{22}{3}\pi \text{ (cm/hr)}$$

□

9. Let $y = f(x) = x - \frac{x^2}{6} - \frac{2\ln x}{3}$, $x > 0$.

Answer the following question and give your reasons (including computations). Put "**None**" in the blank if the item asked does **not** exist.

(a) Find the interval(s) on which f is increasing.

Answer: ☐

(b) Find the local maximal point(s) and minimal point(s) of f, if any.

Answer: local maximal point(s) (x, y) = ☐

local minimal point(s) (x, y) = ☐

(c) Find the interval(s) on which f is concave up.

Answer: ☐

(d) Find the inflection point(s) if any.

Answer: ☐

(e) Sketch the graph of f. Indicate all information in (a)-(d).

Sol.

(a)

$$y' = 1 - \frac{x}{3} - \frac{2}{3x}$$
$$y' > 0 \Rightarrow x^2 - 3x + 2 < 0 \quad (\because x > 0)$$
$$\Rightarrow 1 < x < 2$$

(b)

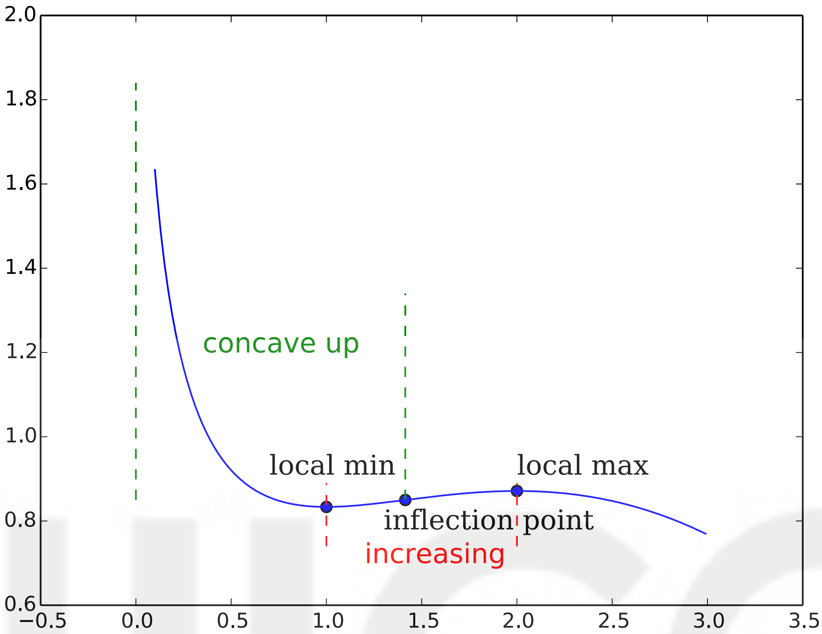
$$\begin{cases} \text{local max} = (2, \frac{4 - 2\ln 2}{3}) \\ \text{local min} = (1, \frac{5}{6}) \end{cases}$$

(c) concave up $\Rightarrow f''(x) > 0$

$$-\frac{1}{3} + \frac{2}{3x^2} > 0$$
$$\frac{2}{x^2} - 1 > 0 \Rightarrow 0 < x < \sqrt{2} \quad (\because x > 0)$$

(d) $f''(x) = 0 \Rightarrow x = \sqrt{2}, -\sqrt{2}(\rightarrow\leftarrow)$, inflection point $= (\sqrt{2}, \sqrt{2} - \frac{1 + \ln 2}{3})$

(e)



concave up
local min
local max
inflection point
increasing

□