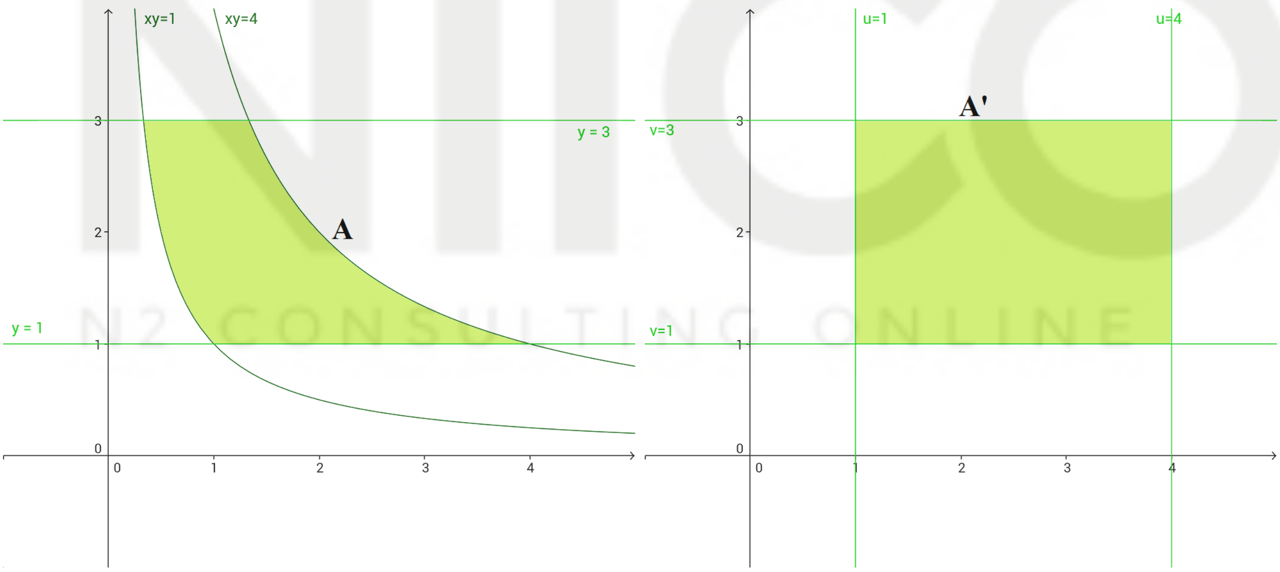
102-2 微甲 07-11 班期末考試題及詳解

1. Evaluate $\displaystyle\iint_A e^{xy}\,dxdy$, where A is the region enclosed by $xy = 1,\ xy = 4,\ y = 1$ and $y = 3$.

Sol.

Let $u = xy,\ v = y \quad \Rightarrow x = \dfrac{u}{y},\ y = v$



$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{y} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{y} = \frac{1}{v}$$

$$\begin{aligned} &\iint_A e^{xy}\,dxdy \\ =&\iint_{A'} e^u \cdot \frac{1}{v}\,dudv \\ =&\int_1^3 \int_1^4 e^u \cdot \frac{1}{v}\,dudv \end{aligned}$$

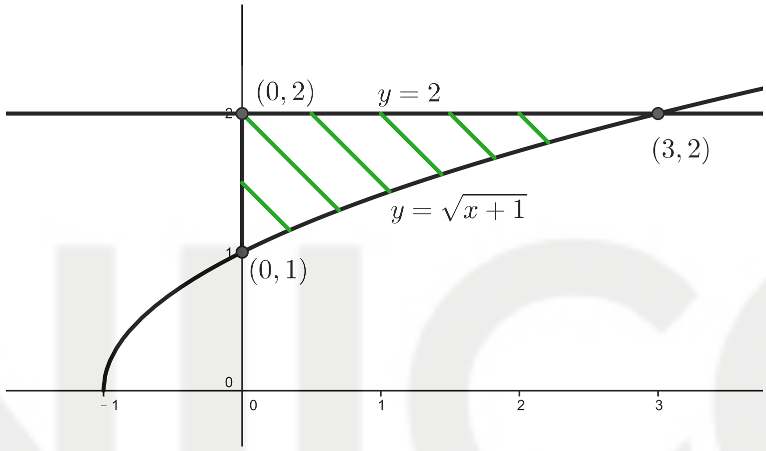
$$= (e^4 - e)\ln 3$$

□

2. Sketch the region of integration and evaluate the integral $\int_0^3 \int_{\sqrt{x+1}}^2 e^{\frac{x}{y+1}}\, dydx$.

Sol.

The region of integration is shown in the figure.



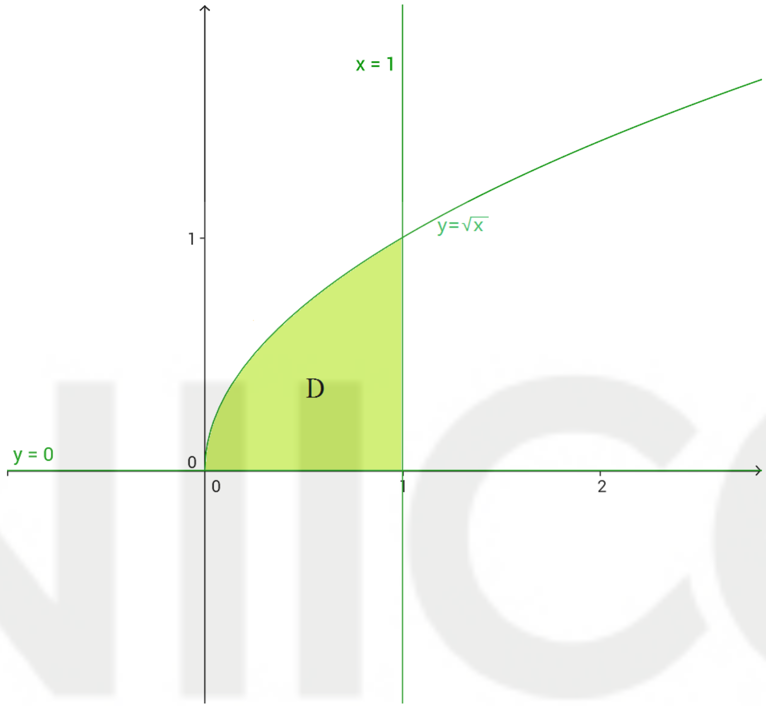
$$\begin{aligned}
&\int_0^3 \int_{\sqrt{x+1}}^2 e^{\frac{x}{y+1}}\, dydx \\
&= \int_1^2 \int_0^{y^2-1} e^{\frac{x}{y+1}}\, dxdy \quad (y = \sqrt{x+1} \Rightarrow x = y^2 - 1) \\
&= \int_1^2 (y+1) e^{\frac{x}{y+1}} \Big|_0^{y^2-1} dy \\
&= \int_1^2 (y+1)(e^{y-1} - 1) dy \\
&= \int_1^2 y \cdot e^{y-1} dy + \int_1^2 e^{y-1} dy - \int_1^2 (y+1) dy \\
&= y e^{y-1} \Big|_1^2 - \int_1^2 e^{y-1} dy + \int_1^2 e^{y-1} dy - (\frac{y^2}{2} + y)\Big|_1^2 \\
&= 2e - 1 - \frac{5}{2} = 2e - \frac{7}{2}
\end{aligned}$$

□

3. Let D be the bounded region in the first quadrant enclosed by $y = 0,\ x = 1$, and $y = \sqrt{x}$ with positively oriented boundary C (i.e. counter clockwise). Evaluate

$$\oint_C \left[9x^2y(x^3+1)^{\frac{1}{2}} - xy^2(x^3+1)^{\frac{3}{2}}\right]dx + \left[2(x^3+1)^{\frac{3}{2}} + 2(y^3+1)^{\frac{3}{2}}\right]dy.$$

Sol.



Let $P(x,y) = 9x^2y(x^3+1)^{\frac{1}{2}} - xy^2(x^3+1)^{\frac{3}{2}}$ and $Q(x,y) = 2(x^3+1)^{\frac{3}{2}} + 2(y^3+1)^{\frac{3}{2}}$

$$\frac{\partial Q}{\partial x} = 9x^2(x^2+1)^{\frac{1}{2}}, \qquad \frac{\partial P}{\partial y} = 9x^2(x^2+1)^{\frac{1}{2}} - 2xy(x^3+1)^{\frac{3}{2}}$$

By Green Theorem:

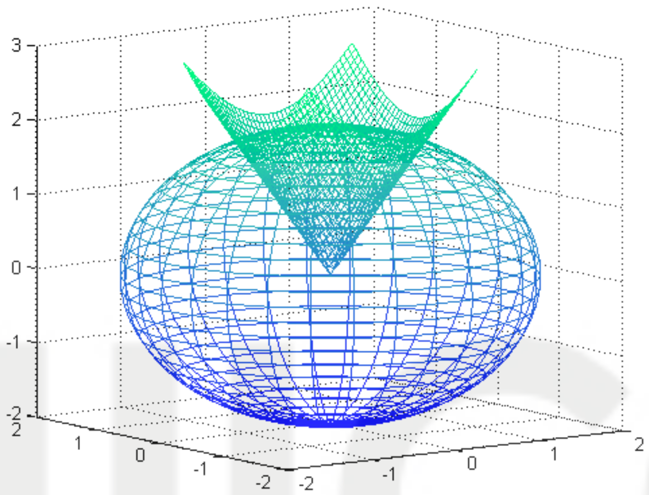
$$\begin{aligned}
&\oint_C Pdx + Qdy \\
&= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \int_0^1 \int_0^{\sqrt{x}} 2xy(x^3+1)^{\frac{3}{2}} dydx \\
&= \int_0^1 x^2(x^3+1)^{\frac{3}{2}} dx \\
&= \frac{2}{15}(x^3+1)^{\frac{5}{2}}\Big|_0^1 \\
&= \frac{2}{15}(2^{\frac{5}{2}} - 1)
\end{aligned}$$

□

4. Evaluate the triple integral $\iiint_E xyz\ dV$ with

$$E = \left\{0 \le x \le \sqrt{4-y^2},\ 0 \le y \le 2,\ \sqrt{x^2+y^2} \le z \le \sqrt{8-x^2-y^2}\right\}.$$

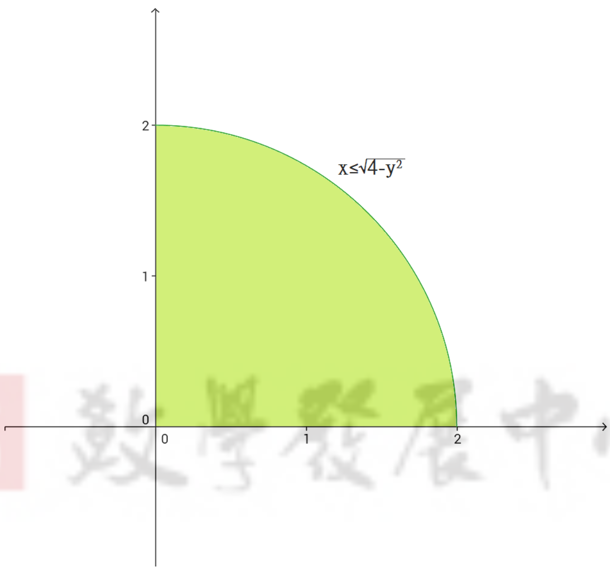
Sol.



(Method 1) Integrate directly.

$$\iiint_E xyz\ dV = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} xyz\ dzdxdy$$

(Method 2) Integrate in cylindrical coordinate.



$$\iiint_E xyz\ dV = \int_0^{\frac{\pi}{2}}\int_0^2\int_r^{\sqrt{8-r^2}} (r\cos\theta)(r\sin\theta)z\ rdzdrd\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_0^2 (4-r^2)r^3\cos\theta\sin\theta\ drd\theta$$

(Method 3) Integrate in polar coordinate.

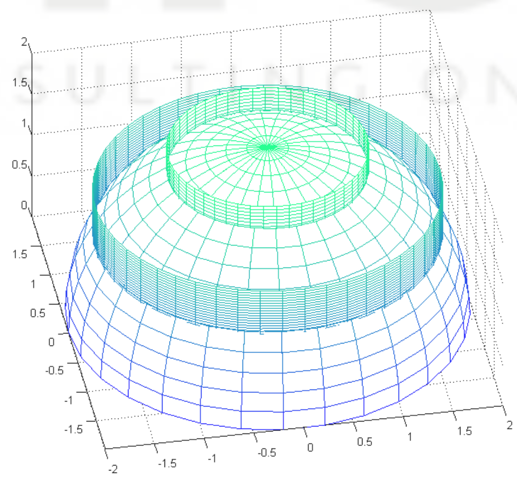
$$\begin{cases} \sqrt{x^2+y^2} = z \\ \sqrt{8-x^2-y^2} = z \end{cases} \Rightarrow \begin{cases} \rho\sin\phi = \rho\cos\phi \\ 8-\rho^2\sin^2\phi = \rho^2\cos^2\phi \end{cases}$$
$$\Rightarrow \begin{cases} \phi = \dfrac{\pi}{4} \\ \rho = 2\sqrt{2} \end{cases}$$

$$\iiint_E xyz\ dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}} (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\ \rho^2 d\rho d\phi d\theta$$

The value of the integral is $\frac{8}{3}$. □

5. Find the area of the surface $\{x^2+y^2+z^2=4,\ 1 \le x^2+y^2 \le 3,\ z \ge 0\}$.

Sol.

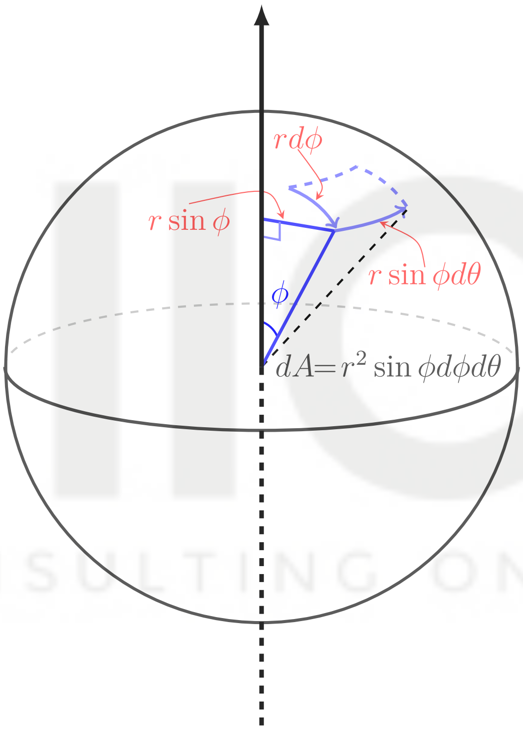


(Method 1)

$$A(S) = \iint_S dS$$
$$= \iint_{1\le x^2+y^2\le 3} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA \quad (z=\sqrt{4-x^2-y^2})$$

$$= \iint\limits_{1\le x^2+y^2\le 3} \sqrt{1+\left(\frac{y^2}{4-x^2-y^2}\right)+\left(\frac{x^2}{4-x^2-y^2}\right)}dA$$
$$= \iint\limits_{1\le x^2+y^2\le 3} \frac{2}{\sqrt{4-x^2-y^2}}dA$$
$$= \int_0^{2\pi}\int_1^{\sqrt{3}} \frac{2}{\sqrt{4-r^2}}rdrd\theta$$
$$= 4\pi(\sqrt{3}-1)$$

(Method 2)



$$A(S) = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r^2\sin\phi\ d\phi d\theta$$
$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\sin\phi\ d\phi d\theta$$
$$= 4\pi(\sqrt{3}-1)$$

□

6. Let S be the part of the sphere $x^2+y^2+(z-2)^2 = 8$ that lies above the xy-plane and that has outward normal (i.e. with $\mathbf{k}$-component ≥ 0). Let $\mathbf{F}(x,y,z) = \langle -y^3\cos xz, x^3e^{yz}, -e^{xyz}\rangle$. Find $\displaystyle\iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S}$.

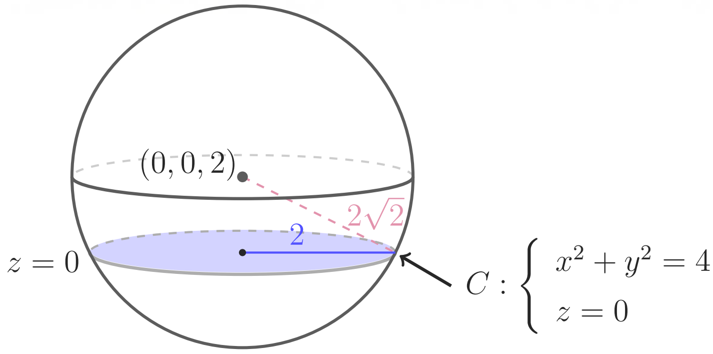
Sol.

(Method 1)

$$\begin{aligned}
\mathbf{r}(t) &= \langle 2\cos t, 2\sin t, 0\rangle \\
\mathbf{r}'(t) &= \langle -2\sin t, 2\cos t, 0\rangle \\
d\mathbf{r} &= \frac{d\mathbf{r}}{dt}dt, \quad 0 \leq t \leq 2\pi \\
\iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S} &= \oint_C \mathbf{F}\cdot d\mathbf{r} \\
&= \oint_C \mathbf{F}\cdot \frac{d\mathbf{r}}{dt}dt \\
&= \oint_C (-8\sin^3 t, 8\cos^3 t, -1)\cdot(-2\sin t, 2\cos t, 0)dt \\
&= 16\int_0^{2\pi} (\sin^4 t + \cos^4 t)dt = 24\pi
\end{aligned}$$

(Method 2)

$$S : \begin{cases} x^2 + y^2(z-2)^2 = 8 \\ z \geq 0 \end{cases} \qquad D : \begin{cases} x^2 + y^2 \leq 4 \\ z = 0 \end{cases}$$



$$\begin{aligned}
\iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_D \text{curl } \mathbf{F}\cdot d\mathbf{D} \\
&= \iint_D \text{curl } \mathbf{F}\cdot (0,0,1)dS \\
&= \iint_D \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \text{F}_x & \text{F}_y & \text{F}_z \end{vmatrix}\cdot (0,0,1)dD
\end{aligned}$$

$$
\begin{aligned}
&= \iint_D \left(\frac{\partial}{\partial x}(x^3 e^{yz}) \Big|_{z=0} \right) - \left(\frac{\partial}{\partial y}(-y^3 \cos xz) \Big|_{z=0} \right) dD \\
&= \iint_D (3x^2 + 3y^2) dD \\
&= \int_0^{2\pi} \int_0^2 3r^2 \; r dr d\theta = 24\pi
\end{aligned}
$$

□

7. (a) Find a scalar function $f(x, y, z)$ such that $\nabla f = \sin y \,\mathbf{i} + x \cos y \,\mathbf{j} - \sin z \,\mathbf{k}$.

   (b) Find the line integral $\int_C \sin y dx + x \cos y dy + (y - \sin z) dz$, where $C : \mathbf{r}(t) = \left\langle t, \frac{\pi}{2} \cos t, \frac{\pi}{2} \sin t \right\rangle, \; 0 \le t \le \pi$.

Sol.

(a)

$$
\begin{aligned}
\nabla f &= \sin y \,\mathbf{i} + x \cos y \,\mathbf{j} - \sin z \,\mathbf{k} \\
&\Rightarrow f = x \sin y + h(y, z) \\
\frac{\partial f}{\partial y} &= x \cos y + \frac{\partial}{\partial y} h(y, z) = x \cos y \\
&\Rightarrow h(y, z) = Cg(z) = g(z) \quad \text{(We let } C = 1) \\
&\Rightarrow f = x \sin y + g(z) \\
\frac{\partial f}{\partial z} &= g'(z) = -\sin z \quad \Rightarrow \; g(z) = \cos z + C \\
& \therefore f = x \sin y + \cos z + C
\end{aligned}
$$

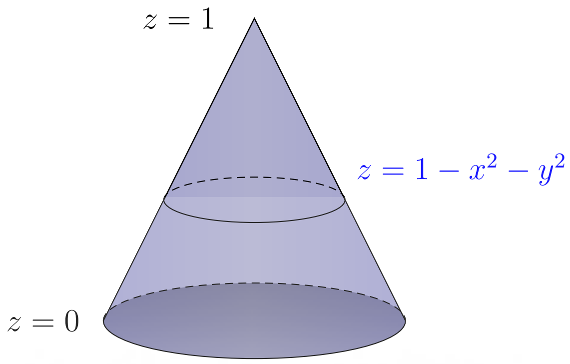
(b)

$$
\begin{aligned}
& \int_C \sin y dx + x \cos y dy + (y - \sin z) dz \\
&= \int \nabla f \cdot d\mathbf{r} + \int y dz \\
&= [f(\mathbf{r}(\pi)) - f(\mathbf{r}(0))] + \int \frac{\pi}{2} \cos t \cdot (\frac{\pi}{2} \cos t \; dt) \\
&= f(\pi, -\frac{\pi}{2}, 0) - f(0, \frac{\pi}{2}, 0) + \frac{\pi^2}{4} \int_0^{\pi} \frac{1 + \cos 2t}{2} dt \\
&= -\pi + \left(\frac{\pi}{2} \right)^3
\end{aligned}
$$

□

8. Let $\mathbf{F}= \langle 3xy^2, y^3, e^{x^2+y^2}\rangle$. Let S be the part of the surface $z = 1 - x^2 - y^2$ that lies above xy-plane oriented upwards(that is, with normal having $\mathbf{k}$-component ≥ 0). Calculate the flux $\int_S \mathbf{F} \cdot dS$ of $\mathbf{F}$ across S. Note that S is not closed.

Sol.



(Method 1)

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(r(x,y)) \cdot (r_x \times r_y) dA$$

$$r(x,y) = (x, y, 1 - x^2 - y^2)$$
$$r_x \times r_y = (1, 0, -2x) \times (0, 1, -2y) = (2x, 2y, 1)$$

$$\begin{aligned} \therefore \int_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D 6x^2y^2 + 2y^4 + e^{x^2+y^2} dA \\ &= \int_0^{2\pi} \int_0^1 \left(6(r\cos\theta)^2(r\sin\theta)^2 + 2(r\sin\theta)^4 + e^{r^2}\right) r dr d\theta \\ &= \pi\left(e - \frac{1}{2}\right) \end{aligned}$$

(Method 2) Let S_1 be the lateral wall of S, and S_2 be the bottom surface of S.

$$\iiint \nabla \cdot \mathbf{F} dV = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S}_1 + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}_2$$

$$\begin{aligned} \therefore \iint_{S_1} \mathbf{F} \cdot d\mathbf{S}_1 & \\ &= \iiint (\nabla \cdot \mathbf{F}) dV - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}_2 \\ &= \iiint 6y^2 dV - \iint \mathbf{F} \cdot (0, 0, -1) d\mathbf{S}_2 \end{aligned}$$

$$
\begin{aligned}
&= \iiint 6y^2 dV - \iint_D -e^{x^2+y^2} dA \\
&= \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6(r\sin\theta)^2 r dz dr d\theta + \int_0^{2\pi} \int_0^1 e^{r^2} r dr d\theta \\
&= \pi \left(e - \frac{1}{2} \right)
\end{aligned}
$$

□