102-2 微甲 07-11 班期中考試題及詳解

1. Find the values of ρ for the convergence of the series below

   (a) $\sum_{n=0}^{\infty} e^{n(\rho^2-\rho-2)}$,

   (b) $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}-1}{n^{\rho}}$.

Sol.

(a)

$$\begin{aligned} e^{\rho^2-\rho-2} < 1 &\Rightarrow \rho^2-\rho-2<0 \\ &\Rightarrow -1<\rho<2 \end{aligned}$$

(b)

$$e^{\frac{1}{n}}-1 = \left(1+\frac{1}{n}+\frac{\left(\frac{1}{n}\right)^2}{2!}+\cdots\right)-1 \approx \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{\frac{e^{\frac{1}{n}}-1}{n^{\rho}}}{\frac{1}{n^{\rho+1}}} = \lim_{n\to\infty} \frac{e^{\frac{1}{n}}-1}{\frac{1}{n}} = \lim_{t\to 0} \frac{e^t-1}{t} = \lim_{t\to 0} e^t = 1$$

$\therefore$ By limit comparision test, $\langle \frac{e^{\frac{1}{n}-1}}{n^{\rho}} \rangle$ and $\langle \frac{1}{n^{\rho+1}} \rangle$ both converge or both diverge

$\therefore$ $\rho+1>1 \Rightarrow \rho>0$

□

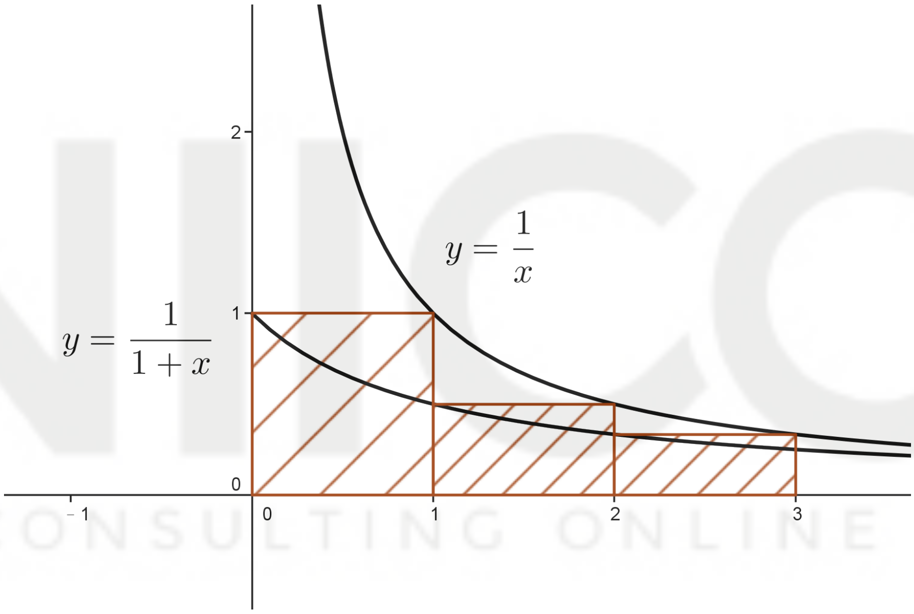
2. (a) Prove $\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln n$.

(b) Test for convergence of $\sum_{n=1}^{\infty} a_n$, where $a_n = \dfrac{1}{1 + \frac{1}{2} + \cdots + \frac{1}{n}}$.

Sol.

(a)

$$\int_0^n \frac{1}{1+x}dx < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{1}{x}$$
$$\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln(n)$$



(b)

$$a_n = \frac{1}{1 + \frac{1}{2} + \cdots + \frac{1}{n}} > \frac{1}{1+\ln(n)} > \frac{1}{1+n}$$
$$\lim_{n\to\infty} \frac{\frac{1}{1+n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{1+n} = 1, \text{ and } \sum \frac{1}{n} \text{ diverges}$$

by comparison test

$$\sum \frac{1}{1+n} \text{ diveges } \Rightarrow \sum a_n \text{ diverges}$$

or

$$\because \int_1^{\infty} \frac{1}{1+x}dx = \infty \Rightarrow \text{by integral test, } \sum \frac{1}{1+n} \text{ diverges} \Rightarrow \sum a_n \text{ diverges}$$

□

3. Let $f(x,y) = \sin(x-y)e^{-x^2-y^2}$, $P = (\sqrt{2}, \sqrt{2})$.

(a) Find the maximum rate of change of f at P.

(b) Find the direction in which the maximum rate of change occurs.

(c) Find the directional derivative $D_{\mathbf{u}}(P)$, where $\mathbf{u} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Sol.

(a) (b)

$$\begin{aligned}
\nabla f(x,y) &= \Big(\cos(x-y)e^{-x^2-y^2} - 2x\sin(x-y)e^{-x^2-y^2}, \\
&\quad - \cos(x-y)e^{-x^2-y^2} - 2y\sin(x-y)e^{-x^2-y^2}\Big) \\
\nabla f(\sqrt{2}, \sqrt{2}) &= (e^{-4}, -e^{-4}) \\
|\nabla f(\sqrt{2}, \sqrt{2})| &= \sqrt{2}e^{-4} \\
\text{direction} &= (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})
\end{aligned}$$

(c)

$$\begin{aligned}
D_{\mathbf{u}}(P) &= \nabla f(P) \cdot \mathbf{u} \\
&= (e^{-4}, -e^{-4}) \cdot (\frac{1}{2}, \frac{\sqrt{3}}{2}) \\
&= \frac{1-\sqrt{3}}{2e^4}
\end{aligned}$$

□

4. Let $\mathbf{r}(t) = \left\langle \frac{t^4}{2}, t, \frac{4}{5}t^{\frac{5}{2}} \right\rangle$ for $t \geq 0$.

(a) Find the length of the arc $0 \leq t \leq 2$ of $\mathbf{r}(t)$.

(b) Find the curvature $\kappa(t)$.

(c) Find $\mathbf{T}(1)$, $\mathbf{N}(1)$ and $\mathbf{B}(1)$, the principal unit normal vector and the binormal unit vector when $t = 1$ respectively.

Sol.

(a)

$$\mathbf{r}(t) = \left(\frac{t^4}{2}, t, \frac{4}{5}t^{\frac{5}{2}}\right)$$

$$\begin{aligned}
\mathbf{r}'(t) &= (2t^3, 1, 2t^{\frac{3}{2}}) \\
|\mathbf{r}'(t)| &= 1 + 2t^3 \\
L &= \int_0^2 (1+2t^3)dt = 10
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{r}''(t) &= (6t^2, 0, 3t^{\frac{1}{2}}) \\
\mathbf{r}'(t) \times \mathbf{r}''(t) &= (3t^{\frac{1}{2}}, 6t^{\frac{7}{2}}, -6t^2) \\
\kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{3\sqrt{t}}{(2t^3+1)^2}
\end{aligned}$$

(c)

$$\begin{aligned}
&\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{1+2t^3}(2t^3, 1, 2t^{\frac{3}{2}}) \\
&\mathbf{T}(1) = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \\
&\mathbf{T}'(t) = \frac{1}{(1+2t^3)^2}(6t^2, -6t^2, -6t^{\frac{7}{2}} + 3t^{\frac{1}{2}}) \\
&\mathbf{T}'(1) = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right),\ \mathbf{N}(1) = \frac{\mathbf{T}'(1)}{|\mathbf{T}'(1)|} = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right) \\
&\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)
\end{aligned}$$

□

5. Find the extreme values of $f(x,y) = x^2y - xy + xy^2$ on $x^2 + xy + y^2 - x - y = 1$.

Sol.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \Rightarrow \begin{cases} y(2x+y-1) = \lambda(2x+y-1) \\ x(x+2y-1) = \lambda(x+2y-1) \\ x^2 + xy + y^2 - x - y = 1 \end{cases}$$

(a) $2x + y - 1 = 0 \Rightarrow (x,y) = \left(\frac{-1}{3}, \frac{5}{3}\right)$ or $(1,-1)$

(b) $2x + y - 1 \neq 0 \Rightarrow \lambda = y$

(i) $x + 2y - 1 = 0 \Rightarrow (x,y) = \left(\frac{5}{3}, \frac{-1}{3}\right)$ or $(1,-1)$

(ii) $x + 2y - 1 \neq 0 \Rightarrow \lambda = x = y \Rightarrow (x,y) = \left(\frac{-1}{3}, \frac{-1}{3}\right)$ or $(1,1)$

We find that $\begin{cases} f(1,-1) = f(-1,1) = f(1,1) = 1 & \boxed{\text{Max}} \\ f\left(\frac{-1}{3}, \frac{5}{3}\right) = f\left(\frac{5}{3}, \frac{-1}{3}\right) = f\left(\frac{-1}{3}, \frac{-1}{3}\right) = \frac{-5}{27} & \boxed{\text{Min}} \end{cases}$

□

6. Find the local maximum, and local minimum values and saddle point(s) of $f(x,y) = y^3 + 3x^2y - 3x^2 - 3y^2 + 3$.

Sol.

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \ \Rightarrow \ 6xy - 6x = 0 \\ \dfrac{\partial f}{\partial y} = 0 \ \Rightarrow \ 3y^2 + 3x^2 - 6y = 0 \end{cases}$$

$$\Rightarrow \begin{array}{l} \text{(a) } x = 0 \ \Rightarrow \ y = 0, 2 \\ \text{(b) } y = 1 \ \Rightarrow \ x = -1, 1 \end{array}$$

we have 4 critical points

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = (6y-6)^2 - (6x)^2 \qquad (f_{xy} = 6x)$$

$$\begin{array}{lll} D(0,0) = 36 > 0 \quad f_{xx} = -6 < 0 & \Rightarrow & \text{max} \\ D(0,2) = 36 > 0 \quad f_{xx} = 6 > 0 & \Rightarrow & \text{min} \\ D(-1,1) = -36 < 0, \ D(1,1) = -36 < 0 & \Rightarrow & \text{saddle point} \end{array}$$

□

7. (a) Find the radius of convergence and the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}}$.

   (b) Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}}$ when the power series is convergent. Evaluate $f^{(3)}(1)$.

Sol.

(a)

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\sqrt{n}}{\sqrt{n+1}}\right| \left|\frac{x-1}{-2}\right| = \left|\frac{x-1}{-2}\right| < 1$$

$$\Rightarrow |x-1| < 2$$

check end points

(i) $x = 3 \ \Rightarrow \ \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges $\left(\because \begin{array}{l} ① \ a_{n+1} < a_n \quad ② \ \lim\limits_{n\to\infty} a_n = 0 \\ ③ \text{‘}a_n \text{ is alternate series} \end{array} \right)$

(ii) $x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (by p-series theorem)

So the radius of convergence is $(-1, 1]$.

(b)

$$f(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$n = 3 \to \frac{1}{(-2)^3\sqrt{3}} = \frac{f^{(3)}(1)}{3!}$$

$$\therefore\ f^{(3)}(1) = \frac{3!}{-8\sqrt{3}} = -\frac{\sqrt{3}}{4}$$

□

8. (a) Write down the general terms the MacLaurin series of $\sin x$ and $\sin^{-1} x$.

   (b) Find their radii of convergence.

   (c) Find $\lim_{x\to 0} \frac{\sin x \cdot \sin^{-1} x - x^2}{x^6}$.

Sol.

(a) (b)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n+1}$$

$$\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} (-t^2)^k dt$$

$$= \sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} (-1)^k \frac{1}{2k+1} x^{2k+1}$$

(c)

$$\lim_{x\to 0} \frac{\sin x \cdot \sin^{-1} x - x^2}{x^6}$$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\cdots\right)\left(x + \frac{x^3}{6} + \frac{3}{40}x^5\cdots\right) - x^2}{x^6}$$

$$= \frac{3}{40} - \frac{1}{36} + \frac{1}{120} = \frac{1}{18}$$

□