90學年度轉學考試試題：工程數學

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Solve problem $1-4$ below.

1. $y'=\dfrac{xy+3x-y-3}{xy-2x+4y-8}$.

Solution: 藉由因式分解可得

$$y'=\frac{(x-1)(y+3)}{(x+4)(y-2)}$$

藉由移項來施行分離變數法

$$\frac{y-2}{y+3}dy=\frac{x-1}{x+4}dx$$

即

$$\left(1-\frac{5}{y+3}\right)dy=\left(1-\frac{5}{x+4}\right)dx$$

同取不定積分可得

$$y-5\ln|y+3|=x-5\ln|x+4|+C$$

□

2. $5y''+y'=-6x,\ y(0)=0,\ y'(0)=-10.$

Solution: 兩邊同取不定積分可得

$$5y'+y=-3x^2+C_1$$

其中C_1爲積分常數，利用初始條件可決定出$C_1=-50$。因此可得

$$y'+\frac{1}{5}y=-\frac{3}{5}x^2-10$$

運用積分因子法，取$\mu=e^{\int\frac{1}{5}dx}=e^{x/5}$爲積分因子，兩邊同乘$\mu$後可得

$$\left(e^{x/5}y\right)'=e^{x/5}y'+\frac{e^{x/5}}{5}y=-\frac{3}{5}x^2e^{x/5}-10e^{x/5}$$

如此再次同取不定積分可得

$$e^{x/5}y=-3x^2e^{x/5}+30xe^{x/5}-200e^{x/5}+C_2$$

因此可得

$$y=-3x^2+30x-200+C_2e^{-x/5}$$

藉由初始條件可解得$C_2=200$，因此所求之函數爲

$$y=-3x^2+30x-200+200e^{-x/5}$$

□

3. $x^3y^{(3)} - 3x^2y'' + 6xy' - 6y = 3 + \ln(x^3)$.
Solution: 令$t = \ln(x)$，如此

$$\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = \frac{1}{x}\frac{dy}{dt}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{dy}{dx} = \frac{dt}{dx}\cdot\frac{d}{dt}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt} \\
\frac{d^3y}{dx^3} &= \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{dt}{dx}\cdot\frac{d}{dt}\left(\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right) \\
&= \frac{1}{x}\cdot\left[-\frac{2}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) + \frac{1}{x^2}\left(\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2}\right)\right] \\
&= \frac{1}{x^3}\frac{d^3y}{dt^3} - \frac{3}{x^3}\frac{d^2y}{dt^2} + \frac{2}{x^3}\frac{dy}{dt}
\end{aligned}$$

據此原本的微分方程可改寫爲

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = 3 + 3t$$

此方程的齊次微分方程所對應的特徵方程爲

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

即爲$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$，因此$\lambda = 1, 2, 3$，故$y_c = c_1e^t + c_2e^{2t} + c_3e^{3t}$；現在考慮特解$y_p$符合

$$\frac{d^3y_p}{dt^3} - 6\frac{d^2y_p}{dt^2} + 11\frac{dy_p}{dt} - 6y_p = 3 + 3t$$

可猜測$y_p = A + Bt$，代入後有

$$(11B - 6A) - 6Bt = 3 + 3t$$

如此比較係數可得$A = -\frac{17}{12}$、$B = -\frac{1}{2}$。因此$y_p = -\frac{17}{12} - \frac{1}{2}t$。因此通解$y$爲

$$y = y_c + y_p = c_1e^t + c_2e^{2t} + c_3e^{3t} - \frac{17}{12} - \frac{1}{2}t$$

將變數換爲x，即有

$$y = c_1x + c_2x^2 + c_3x^3 - \frac{17}{12} - \frac{1}{2}\ln(x)$$

□

4. $\begin{cases} ku_{xx} = u_t, 0 < x < L, t > 0 \\ u(0,t) = 0, u(L,t) = 0 \\ u(x,0) = x(L-x) \end{cases}$

Solution: 運用分離變數法，設$u(x,t) = X(x)T(t)$，如此有

$$\begin{cases} X(x)T'(t) = kX''(x)T(t) \\ X(0)T(t) = X(L)T(t) = 0 \end{cases}$$

由於我們要找非零解，因此邊界條件得到$X(0) = X(L) = 0$。另一方面，方程本身可改寫爲$\dfrac{X''(x)}{X(x)} = \dfrac{T'(t)}{kT(t)}$，左式爲$x$的函數而右式爲$t$函數，兩者相等表示此爲一常量，命之爲$-\lambda$，因此有

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ T'(t) + \lambda kT(t) = 0 \end{cases}$$

當$\lambda < 0$時，記$\lambda = -\mu^2 < 0$，則能解出$X(x) = c_1e^{\mu x} + c_2e^{-\mu x}$，接著根據邊界條件$X(0) = X(L) = 0$可解得$c_1 = c_2 = 0$，即$X(x)$恆爲零，因此予以排除；當$\lambda = 0$時能解出$X(x) = A + Bx$，而根據邊界條件則能給出$A = B = 0$，即$X(x)$恆爲零，因此予以排除。

當$\lambda > 0$時，記$\lambda = \eta^2 > 0$，則能解出$X(x) = C_1\cos(\eta x) + C_2\sin(\eta x)$。由$X(0) = 0$可得$C_1 = 0$，進一步利用$X(L) = 0$則得$C_2\sin(\eta L) = 0$。由於我們希望$C_2 \neq 0$，因此$\sin(\eta L) = 0$，故$\eta$可爲$\eta = \dfrac{n\pi}{L}$。因此我們可以得到

$$\begin{cases} X_n(x) = A_n\sin\left(\dfrac{n\pi}{L}x\right) \\ T_n(t) = B_n\exp\left(-\dfrac{n^2\pi^2k}{L^2}t\right) \end{cases}$$

運用解的疊加性可得

$$u(x,t) = \sum_{n=1}^{\infty} K_n\sin\left(\frac{n\pi}{L}x\right)\exp\left(-\frac{n^2\pi^2k}{L^2}t\right)$$

根據邊界條件可得

$$x(L-x) = u(x,0) = \sum_{n=1}^{\infty} K_n\sin\left(\frac{n\pi}{L}x\right)$$

兩邊同乘以$\sin\left(\dfrac{m\pi}{L}x\right)$後在$[0,L]$取定積分，如此有

$$L\int_0^L x\sin\left(\frac{m\pi}{L}x\right)dx - \int_0^L x^2\sin\left(\frac{m\pi}{L}x\right)dx = K_m\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx$$

即有

$$\frac{2L^3(1-\cos(m\pi))}{m^3\pi^3} = \frac{K_mL}{2}$$

因此$K_n = \dfrac{4L^2\left(1-(-1)^n\right)}{n^2\pi^2}$，因此

$$u(x,t) = \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\sin\left(\frac{n\pi}{L}x\right)\exp\left(-\frac{n^2\pi^2k}{L^2}t\right)$$

□

5. For BVP $y'' + \lambda y = 0$; $y(-3\pi) = y(3\pi)$, $y'(-3\pi) = y'(3\pi)$, find the **eigenvalues**. Corresponding to each eigenvalue, find an **eigenfunction**. Also prove the **orthogonality**.

Solution: 若$\lambda > 0$，記$\lambda = \mu^2 > 0$，則能解得$y(x) = c_1e^{\mu x} + c_2e^{-\mu x}$。由$y(-3\pi) = y(3\pi)$能得$c_1 = c_2$，而$y'(-3\pi) = y'(3\pi)$可得$c_1 = -c_2$，因此$c_1 = c_2 = 0$，故當$\lambda < 0$時不存在特徵函數。

若$\lambda = 0$，則$y(x) = A + Bx$。由$y(-3\pi) = y(3\pi)$，則有$B = 0$，此時特徵函數y可為任意常數函數。

若$\lambda > 0$，則記$\lambda = \eta^2 > 0$，此能可解得$y(x) = C_1\cos(\eta x) + C_2\sin(\eta x)$。由$y(-3\pi) = y(3\pi)$可得

$$C_2\sin(3\pi\eta) = 0$$

因此$C_2 = 0$或$\eta = \dfrac{n}{3}$。當$C_2 = 0$時，則由$y'(-3\pi) = y'(3\pi)$可得$C_1\sin(3\pi\eta) = 0$。又希望能找到特徵函數，因此要求$\sin(3\pi\eta) = 0$，因而有$\eta = \dfrac{n}{3}$。故當$\lambda = \dfrac{n^2}{9}$時有特徵函數$y_n = C_1\sin\left(\dfrac{n}{3}x\right) + C_2\cos\left(\dfrac{n}{3}x\right)$

因此總結如下：$\lambda = \dfrac{n^2}{9}$為特徵值，其中n為非負整數，其所應的特徵函數為

$$y_n = C_1\sin\left(\frac{n}{3}x\right) + C_2\cos\left(\frac{n}{3}x\right)$$

再者，根據傅立葉級數的知識或一般微積分的計算可知道這些特徵函數兩兩相異者相乘計算定積分會獲得0，而相同者相乘的積分則非零，這表明這些特徵函數具有正交性(orthogonality)

□

6. Find the Fouruer series of $f(x)=\begin{cases}0, & -\pi<x<0\\ 1, & 0\le x<\pi\end{cases}$ on the given interval.

Solution: 將f表爲如下的傅立葉級數之形式：

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

其中

$$\begin{aligned}a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx \qquad (n=0,1,2,\cdots)\\ b_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx \qquad (n=1,2,\cdots)\end{aligned}$$

先計算a_0如下

$$a_0=\frac{1}{\pi}\int_0^{\pi}1dx=1$$

接著計算a_n（$n\neq 0$）如下

$$a_n=\frac{1}{\pi}\int_0^{\pi}\cos nx dx=\frac{1}{n\pi}\sin nx\Big|_0^{\pi}=0 \text{ if } n\neq 0$$

最後計算b_n如下

$$b_n=\frac{1}{\pi}\int_0^{\pi}\sin nx dx=-\frac{1}{n\pi}\cos nx\Big|_0^{\pi}=\frac{1-(-1)^n}{n\pi}=\begin{cases}\dfrac{2}{(2k+1)\pi}, & \text{if } n=2k+1\\ 0, & \text{if } n=2k\end{cases}$$

因此f的傅立葉級數可表示如下

$$\frac{1}{2}+\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{\sin\left[(2k+1)x\right]}{2k+1}$$

□

7. For the matrix A, $A=\begin{bmatrix}2&0&1\\0&0&-2\\0&2&0\end{bmatrix}$. Solve $AX=B$, $B=\begin{bmatrix}e^{-t}\\2+t\\0\end{bmatrix}$.

Solution: 考慮$X=\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\end{bmatrix}$，如此$AX=B$可表達爲

$$\begin{cases}2x_1+x_3=e^{-t}\\ -2x_3=2+t\\ 2x_2=0\end{cases}$$

因此可解得

$$\begin{cases} x_1(t) = \dfrac{1}{2} + \dfrac{t}{4} + \dfrac{e^{-t}}{2} \\ x_2(t) = 0 \\ x_3(t) = -1 - \dfrac{t}{2} \end{cases}$$

□

8. $y'' - e^x y' + 2y = 1$, $y(0) = -3$, $y'(0) = 0$, 寫出其級數解之前五項。

Solution: 設$y(x) = \sum_{n=0}^{\infty} a_n x^n$，由$y(0) = -3$可得$a_0 = -3$；由$y'(0) = 0$可得$a_1 = 0$。

在原微分方程中代入$x = 0$可得$y''(0) = 7$，因此$a_2 = \dfrac{f''(0)}{2!} = \dfrac{7}{2}$。

現在對微分方程再進行一次微分可得$y''' - e^x y'' - e^x y' + 2y' = 0$，接著代入$x = 0$有

$$y'''(0) - y''(0) + y'(0) = 0$$

因此有$y'''(0) = y''(0) = 7$，故$a_3 = \dfrac{f'''(0)}{3!} = \dfrac{7}{6}$。

再微分一次可得$y^{(4)} - e^x y''' - 2e^x y'' - e^x y' + 2y'' = 0$，取$x = 0$代入可得

$$y^{(4)}(0) - y'''(0) - y'(0) = 0$$

因此有$y^{(4)}(0) = 7$，因此$a_4 = \dfrac{f^{(4)}(0)}{4!} = \dfrac{7}{24}$。

故級數解的前五項爲

$$-3 + 0x + \frac{7}{2}x^2 + \frac{7}{6}x^3 + \frac{7}{24}x^4$$

□

Problem 9 and 10, find the indicated expression for the vector field $\mathbf{F} = x^2 y\mathbf{i} + xy^2\mathbf{j} + 2xyz\mathbf{k}$.

9. $\nabla \cdot \mathbf{F}$.

Solution: 根據散度的定義計算可得

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\left(x^2 y\right) + \frac{\partial}{\partial y}\left(xy^2\right) + \frac{\partial}{\partial z}(2xyz) = 2xy + 2xy + 2xy = 6xy$$

□

10. $\nabla \cdot (\nabla \times \mathbf{F})$.

Solution: 先計算$\nabla \times \mathbf{F}$如下

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & xy^2 & 2xyz \end{vmatrix} = 2xz\mathbf{i} - 2yz\mathbf{j} + \left(y^2 - x^2\right)\mathbf{k}$$

接著再計算$\nabla \cdot (\nabla \times \mathbf{F})$可得

$$\nabla \cdot (\nabla \times \mathbf{F}) = \nabla \cdot \left[2xz\mathbf{i} - 2yz\mathbf{j} + \left(y^2 - x^2\right)\mathbf{k}\right] = 2z - 2z + 0 = 0$$

□