101 微甲 07-11 班試題及詳解

101-1 微甲 07-11 班期中考試題及詳解

1. Find $\lim\limits_{x\to 0^-}$ and $\lim\limits_{x\to 0^+}$ of the following functions:

   (a) $\dfrac{\sin(|x|)}{x}$,

   (b) $\dfrac{\cos x - 1}{\sin(x\sin x)}$,

   (c) $\dfrac{\cos(\sin x) - 1}{\tan^2 x}$.

   Sol.

   (a)

$$\lim_{x\to 0^+}\frac{\sin(|x|)}{x} = \lim_{x\to 0^+}\frac{\sin x}{x} = 1$$
$$\lim_{x\to 0^-}\frac{\sin(|x|)}{x} = \lim_{x\to 0^-}-\frac{\sin x}{x} = -1$$

   (b)

$$\begin{aligned}
&\lim_{x\to 0}\frac{\cos x - 1}{\sin(x\cos x)}\frac{\cos x + 1}{\cos x + 1}\\
=&\lim_{x\to 0}-\frac{\sin^2 x}{\sin(x\sin x)(\cos x + 1)}\\
=&\lim_{x\to 0}-\frac{x\sin x}{\sin(x\sin x)}\frac{\sin x}{x}\frac{1}{\cos x + 1}\\
=&\lim_{x\to 0}-\frac{x\sin x}{\sin(x\sin x)}\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x + 1}\\
=&-1\times 1\times\frac{1}{2} = -\frac{1}{2}
\end{aligned}$$

   (c)

$$\begin{aligned}
&\lim_{x\to 0}\frac{\cos(\sin x) - 1}{\tan^2 x}\\
=&\lim_{x\to 0}\frac{\cos^2(\sin x) - 1}{\tan^2 x(\cos(\sin x) + 1)}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 0} -\frac{\sin^2(\sin x)\cos^2 x}{\sin^2 x(\cos(\sin x)+1)} \\
&= -\lim_{x\to 0}\left(\frac{\sin(\sin x)}{\sin x}\right)^2 \cdot \lim_{x\to 0}\frac{\cos^2 x}{\cos(\sin x)+1} \\
&= -\frac{1}{2}
\end{aligned}
$$

□

2. Show that $\left|\tan\frac{x}{2} - \tan\frac{y}{2}\right| \geq \frac{|x-y|}{2}$ for $x, y \in (-\pi, \pi)$.

Sol.

It's equivalent to show that

$$\frac{|\tan\frac{x}{2} - \tan\frac{y}{2}|}{|x-y|} \geq \frac{1}{2}$$

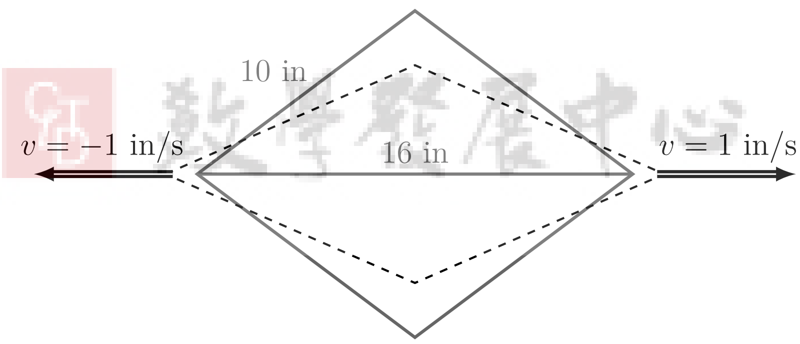
By Mean Value Theorem, $\exists c \in (y, x)$

$$
\begin{aligned}
\to \frac{|\tan\frac{x}{2} - \tan\frac{y}{2}|}{|x-y|} &= \left(\frac{d}{d\theta}\tan\frac{\theta}{2}\right)|_{\theta=c} \\
&= \frac{1}{2}\sec^2\frac{c}{2} \geq \frac{1}{2}
\end{aligned}
$$

□

3. A rhombus (菱形) has sides 10 in. long. Two of its opposite vertices are pulled apart at a rate of 2 in. per second. How fast is the area changing when the vertices being pulled are 16 in apart?

Sol.



$$
x^2 + y^2 = 100, \quad y = \sqrt{100 - x^2}
$$
$$
\frac{dA}{dt} = \frac{d}{dt}\left(\frac{xy}{2} \times 4\right) = \frac{d}{dt}(2xy)
$$

$$
\begin{aligned}
&= \frac{d}{dt}(2x\sqrt{100-x^2}) \\
&= \frac{d}{dx}\left(2x\sqrt{100-x^2}\right)\Big|_{x=8} \cdot \frac{dx}{dt} \\
&= \left(2\sqrt{100-x^2} + 2x\frac{-x}{\sqrt{100-x^2}}\right)\Big|_{x=8} \cdot (1 \text{ in}/s) \\
&= \left(2\times 6 + 2\times 8 \times \frac{-8}{6}\right) = \frac{-28}{3}(\text{in}^2/s)
\end{aligned}
$$

□

4. Let $f(x) = \dfrac{1+\cos x}{1+\sin x}$. Use a differential to estimate $f(44°)$.

Sol.

$$f(x) = \frac{1+\cos x}{1+\sin x}$$

We know that $y_2 = f(x+\Delta x) \approx y_1 + m\Delta x$, where $m = \Delta y/\Delta x$

$$
\begin{aligned}
f(44°) &= f(45° - 1°) \\
&\approx f(45°) + (-1)° f'(45°) \\
f(x) &= \left(\frac{1+\cos x}{1+\sin x}\right)' = \frac{-\sin x - \cos x - 1}{(1+\sin x)^2} \\
f'(45°) &= \frac{-1/\sqrt{2} - 1/\sqrt{2} - 1}{(1+\frac{1}{\sqrt{2}})^2} = 2 - 2\sqrt{2} \\
f(44°) &\approx f(45°) + (-1)° f'(45°) = 1 + \left(2 - 2\sqrt{2}\right)\left(\frac{-\pi}{180}\right) = 1 + \frac{\sqrt{2}-1}{90}\pi
\end{aligned}
$$

□

5. Let $f(x) = \dfrac{(x+1)^2}{x^2+1}$.
   (a) Find f' and f''.
   (b) Find the intervals on which f increases and the intervals on which f decreases. Indicate local extreme values and absolute extreme values.
   (c) Find the intervals on which the graph of f is concave up and the intervals on which the graph f is concave down. Indicate points of inflection.
   (d) Find vertical and horizontal asymptotes if any. Sketch the graph of f.

Sol.

$$f'(x) = \frac{-2(x^2-1)}{(x^2+1)^2}$$

$$f''(x) = \frac{4x(x^2-3)}{(x^2+1)^3}$$

x		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
$f'(x)$	-	-	-	0	+	+	+	0	-	-	-
$f''(x)$	-	0	+	+	+	0	-	-	-	0	+
$f(x)$				0(min)				2(max)			

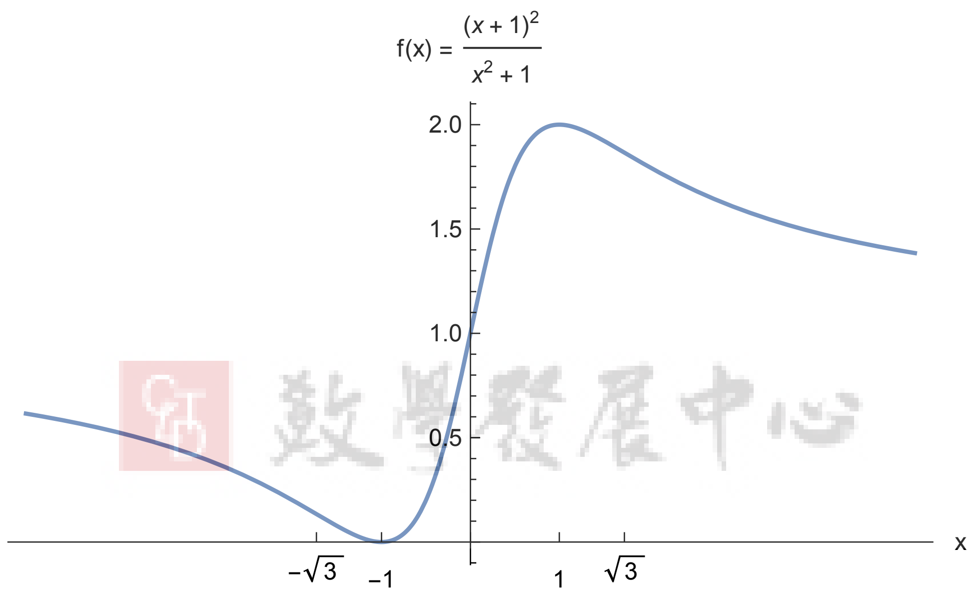
$$f(x) = \frac{(x+1)^2}{x^2+1}$$

$x^2+1>0$, no vertical asymptotes.

$$\lim_{x\to\infty} \frac{(x+1)^2}{x^2+1} = 1$$

$$\lim_{x\to-\infty} \frac{(x+1)^2}{x^2+1} = 1$$

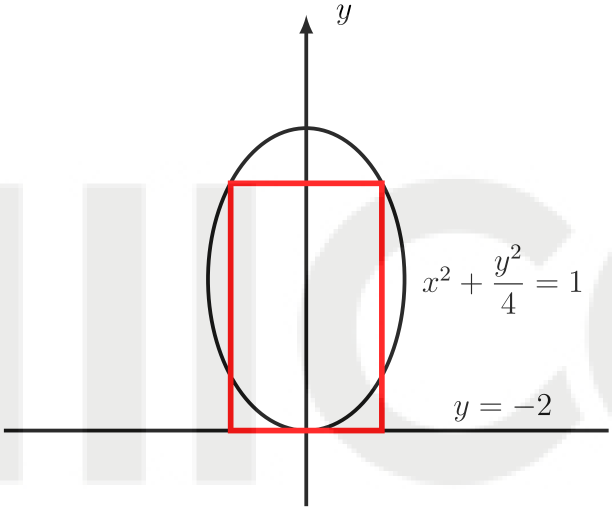
$f(x)$ have horizontal asymptote $y=1$



□

6. Consider all the rectangles with base on line $y = -2$ and with two upper vertices on the ellipse $x^2 + \frac{y^2}{4} = 1$ and symmetric with respect to the y-axis. Find the maximal possible area for such a rectangle.

Sol.



$$A = 2\cos\theta(2\sin\theta - (-2)) = 4\cos\theta(\sin\theta + 1)$$

$$\begin{aligned}\frac{dA}{d\theta} &= -4\sin\theta(\sin\theta + 1) + 4\sin^2\theta \\ &= -4\sin^2\theta - 4\sin\theta + 4(1 - \sin^2\theta) \\ &= -8\sin^2\theta - 4\sin\theta + 4\end{aligned}$$

$$\frac{dA}{d\theta} = 0, \quad 2\sin^2\theta + \sin\theta - 1 = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$$\sin\theta = \frac{1}{2} \text{ or } -1(\times)$$

$$\max = 4 \cdot \frac{\sqrt{3}}{2}\frac{3}{2} = 3\sqrt{3} \quad (\text{when } \theta = \frac{\pi}{6})$$

□

7. Find $f'(2)$ given that $f(x) = \int_{2x}^{x^3-4} \frac{x}{1+\sqrt{t}} dt$.

Sol.

$$
\begin{aligned}
f'(x) &= \left(x \int_{2x}^{x^3-4} \frac{1}{1+\sqrt{t}} dt \right)' \\
&= \int_{2x}^{x^3-4} \frac{dt}{1+\sqrt{t}} dt + x \left(\frac{3x^2}{1+\sqrt{x^3-4}} - \frac{2}{1+\sqrt{2x}} \right) \\
f'(2) &= \int_{4}^{4} \frac{dt}{1+\sqrt{t}} + 2 \left(\frac{12}{1+2} - \frac{2}{1+2} \right) \\
&= \frac{20}{3}
\end{aligned}
$$

□

8. Calculate $\int \frac{\csc^2 2x}{\sqrt{2+\cot 2x}} dx$.

Sol.

Let $2 + \cot 2x = u$, $-2\csc^2 2x dx = du$

$$
\begin{aligned}
\therefore \int \frac{\csc^2 2x}{\sqrt{2+\cot 2x}} dx &= \int \frac{-du}{2\sqrt{u}} \\
&= -\sqrt{u} + C \\
&= -\sqrt{\cot 2x + 2} + C
\end{aligned}
$$

□

101-1 微甲 07-11 班期末考試題及詳解

1. Solve the initial-value problem:

$$\begin{cases} (\sec x)y' + y = (\tan x)e^{\cos x - \sin x}, & 0 \le x < \dfrac{\pi}{2}, \\ y(0) = 0 \end{cases}$$

Sol.

$$y' + (\cos x)y = (\sin x)e^{\cos x - \sin x}$$

$$I(x)(y' + (\cos x)y) = I(x)(\sin x)e^{\cos x - \sin x}$$

$$(I(x)y)' = I(x)y' + I(x)(\cos x)y$$

$$I'(x)y + I(x)y' = I(x)y' + I(x)(\cos x)y$$

$$\frac{I'(x)}{I(x)} = \cos x \Rightarrow \ln|I(x)| = \sin x + C$$

$$I(x) = e^{\sin x} \qquad \text{Let } C = 0$$

$$(e^{\sin x}y)' = e^{\sin x}(\sin x)e^{\cos x - \sin x}$$

$$e^{\sin x}y = \int \sin x e^{\cos x}\,dx$$

$$y = e^{-\sin x}(-e^{\cos x} + C)$$

$$y(0) = 0 \Rightarrow 0 = -e + C$$

$$\therefore C = e$$

$$\therefore y(x) = e^{-\sin x}(-e^{\cos x} + e)$$

□

2. Solve the initial-value problem:

$$\begin{cases} (xy^2 + y^2 + x + 1)dx + (y-1)dy = 0, \\ y(2) = 0 \end{cases}$$

Sol.

$$\begin{aligned}
(x+1)(y^2+1)dx &= (1-y)dy \\
(x+1)dx &= \frac{1-y}{y^2+1}dy \\
\int (x+1)dx &= \int \frac{1-y}{y^2+1}dy \\
\frac{1}{2}x^2 + x &= \int \frac{dy}{y^2+1} - \int \frac{y}{y^2+1}dy \\
&= \tan^{-1} y - \frac{1}{2}\ln(y^2+1) + C \\
y(2) = 0 \Rightarrow \frac{1}{2} \cdot 2^2 + 2 &= C = 4 \\
\frac{1}{2}x^2 + x &= \tan^{-1} y - \frac{1}{2}\ln(y^2+1) + 4
\end{aligned}$$

□

3. Consider the region bounded by the curves:

$$y = \sin\frac{\pi x}{2} \text{ and } y = \frac{6}{x^2+3x+2} \text{ for } 0 \le x \le 1.$$

Note that the two curves meet at $x = 1$. Find the volume of revolving the region about

(a) the y-axis

(b) the x-axis

Sol.

(a)

$$V = 2\pi \int_0^1 x \left(\frac{6}{x^2+3x+2} - \sin\frac{\pi x}{2} \right) dx = 12\pi \ln\frac{9}{8} - \frac{8}{\pi}$$

(b)

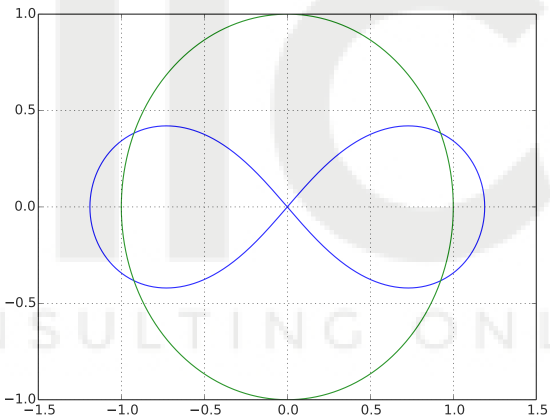
$$V = \int \pi (y_1)^2 dx - \int \pi (y_2)^2 dx$$

$$= \pi \int \left[\left(\frac{6}{x^2+3x+2} \right)^2 - \left(\sin \frac{\pi x}{2} \right)^2 \right] dx$$
$$= \pi \left(\frac{47}{2} + 72 \ln \frac{3}{4} \right)$$

□

4. Find the area both inside $r^2 = 2\cos 2\theta$ and inside $r = 1$.

Sol.



$$\begin{cases} r^2 = 2\cos 2\theta \\ r = 1 \end{cases}$$
$$\Rightarrow \cos 2\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$A = 4\left[\int_0^{\pi/6} \frac{1}{2} r^2 d\theta + \int_{\pi/6}^{\pi/4} \frac{1}{2} \cdot 2\cos 2\theta d\theta \right]$$
$$= 4\left[\frac{1}{2}\frac{\pi}{6} + \frac{1}{2}\sin 2\theta \Big|_{\pi/6}^{\pi/4} \right] = 4\left(\frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4} \right)$$
$$= \frac{\pi}{3} + 2 - \sqrt{3}$$

□

5. Find the area of the surface generated by revolving the curve $r = \sin\theta$, $0 \le \theta \le \frac{\pi}{2}$ about the x-axis.

Sol.

$$A = \int 2\pi y dl = 2\pi \int y\sqrt{dx^2 + dy^2}$$
$$= 2\pi \int y\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}d\theta$$

$$\begin{cases} x = r\cos\theta = \sin\theta\cos\theta \\ y = r\sin\theta = \sin^2\theta \end{cases}, \qquad \begin{cases} x'(\theta) = \cos^2\theta - \sin^2\theta = \cos 2\theta \\ y'(\theta) = 2\sin\theta\cos\theta = \sin 2\theta \end{cases}$$

$$A = \int_0^{\frac{\pi}{2}} 2\pi y\sqrt{(x')^2 + (y')^2}d\theta$$
$$= \int_0^{\frac{\pi}{2}} 2\pi\sin^2\theta d\theta$$
$$= \int_0^{\frac{\pi}{2}} 2\pi\frac{1 - \cos 2\theta}{2}d\theta$$
$$= \frac{\pi^2}{2}$$

□

6. The cycloid is the curve parametrized by $x(\theta) = R(\theta - \sin\theta)$, $y(\theta) = R(1 - \cos\theta)$.

   (a) Find the arc length of the cycloid for $0 \le \theta \le 2\pi$.

   (b) Find the area under the cycloid and above the x-axis for $0 \le \theta \le 2\pi$.

   (c) Find the volume of the solid generated by revolving the region in (b) about the x-axis.

Sol.

(a)

$$l = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}d\theta$$
$$= \int_0^{2\pi} R\sqrt{(1 - \cos\theta)^2 + \sin^2\theta d\theta}$$

$$
\begin{aligned}
&= \int_0^{2\pi} 2R\sin\frac{\theta}{2}d\theta \\
&= 8R
\end{aligned}
$$

(b)

$$
\begin{aligned}
A &= \int_0^{2\pi R} ydx = \int_0^{2\pi} y(\theta)x'(\theta)d\theta \\
&= \int_0^{2\pi} R(1-\cos\theta)R(1-\cos\theta)d\theta = 3\pi R^2
\end{aligned}
$$

(c)

$$
\begin{aligned}
V &= \int_0^{2\pi R} \pi y^2 dx = \int_0^{2\pi} \pi y^2(\theta)x'(\theta)d\theta \\
&= \pi \int_0^{2\pi} [R(1-\cos\theta)]^3 d\theta \\
&= 5\pi^2 R^3
\end{aligned}
$$

□

7. Let $A = \int_0^{\infty} e^{-x^2}dx$. Compute the limit

$$
\lim_{x\to\infty} xe^{x^2}\left(A - \int_0^x e^{-t^2}dt\right).
$$

Sol.

$$
\begin{aligned}
&\lim_{x\to\infty} xe^{x^2}\left(A - \int_0^x e^{-t^2}dt\right) \\
=&\lim_{x\to\infty} \frac{\left(A - \int_0^x e^{-t^2}dt\right)}{x^{-1}e^{-x^2}} \\
=&\lim_{x\to\infty} \frac{-e^{x^2}}{-x^{-2}e^{-x^2} - 2e^{-x^2}} = \lim_{x\to\infty} \frac{1}{2+x^{-2}} \\
=&\frac{1}{2}
\end{aligned}
$$

□

8. Evaluate

(a) $\displaystyle\int_0^1 x\tan^{-1}(x^2)dx$

(b) $\displaystyle\int_0^1 x\left(\tan^{-1}x\right)^2 dx.$

Sol.

(a) By using integration by parts, let $u = \tan^{-1}(x^2)$, $dv = xdx$

$$\int_0^1 x\tan^{-1}\left(x^2\right)dx = \frac{x^2}{2}\tan^{-1}\left(x^2\right)\Big|_0^1 - \int_0^1 \frac{x^2}{2}\frac{2x}{1+x^4}dx$$

$$= \frac{\pi}{8} - \frac{1}{4}\ln(1+x^4)\Big|_0^1$$

$$= \frac{\pi}{8} - \frac{1}{4}\ln 2$$

(b) Let $u = (\tan^{-1}x)^2$, $dv = xdx$

$$\int_0^1 x(\tan^{-1}x)^2dx$$

$$= \frac{x^2}{2}(\tan^{-1}x)^2\Big|_0^1 - \int_0^1 \frac{x^2}{2}2\left(\tan^{-1}x\right)\frac{1}{x^2+1}dx$$

$$= \frac{\pi^2}{32} - \int_0^1 \tan^{-1}x dx + \int_0^1 \tan^{-1}x \cdot \frac{1}{1+x^2}dx$$

$$= \frac{\pi^2}{32} - x\tan^{-1}x\Big|_0^1 + \int_0^1 \frac{x}{1+x^2}dx + \frac{(\tan^{-1}x)^2}{2}\Big|_0^1$$

$$= \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2}\ln(1+x^2)\Big|_0^1 + \frac{\pi^2}{32}$$

$$= \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\ln 2$$

□

9. Evaluate the improper integral $\displaystyle\int_0^2 \frac{1}{x}\sqrt{x(2-x)}dx$.

Sol.

$$\begin{aligned}
\int_0^2 \frac{1}{x}\sqrt{x(2-x)}dx &= \int_0^2 \frac{1}{x}\sqrt{-x^2-2x}dx \\
&= \int_0^2 \frac{1}{x}\sqrt{-(x-1)^2+1}dx \\
&\quad (\text{Let } x-1=\sin\theta,\ dx=\cos\theta d\theta) \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2\theta}{1+\sin\theta}d\theta \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{1-\sin^2\theta}{1+\sin\theta}d\theta \\
&= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} (1-\sin\theta)d\theta \\
&= (\theta+\cos\theta)\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = \pi
\end{aligned}$$

□

10. Find

(a) $\displaystyle\lim_{n\to\infty}\left(\sin\frac{1}{n}\right)^{\frac{1}{n}}$

(b) $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n}$

Sol.

(a)

$$\begin{aligned}
\lim_{n\to\infty}\left(\sin\frac{1}{n}\right)^{\frac{1}{n}} &= \lim_{n\to\infty} e^{\frac{1}{n}\ln\sin\frac{1}{n}} \\
&= \exp\left(\lim_{n\to\infty}\frac{1}{n}\ln\sin\frac{1}{n}\right) \\
&= \exp\left(\lim_{n\to\infty}\frac{\cos\frac{1}{n}}{\sin\frac{1}{n}}\cdot\frac{-1}{n^2}\right) \\
&= 1
\end{aligned}$$

(b)

$$
\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n &= \exp\left(\lim_{n\to\infty} n\ln\left(1+\frac{1}{n^2}\right)\right) \\
&= \exp\left(\lim_{n\to\infty}\frac{\ln(1+\frac{1}{n^2})}{\frac{1}{n}}\right) \\
&= \exp\left(\lim_{n\to\infty}\frac{\frac{-2}{n^3}}{(1+\frac{1}{n^2})\frac{-1}{n^2}}\right) \\
&= \exp\left(\lim_{n\to\infty}\frac{2}{n+\frac{1}{n}}\right) \\
&= e^0 = 1
\end{aligned}
$$

□

101-2 微甲 07-11 班期中考試題及詳解

1. Test the following two series $\sum_{k=1}^{\infty} \frac{\ln k}{k^p}$, where $p = 1$ and $p = 3/2$, for convergence.

Sol.

1) when $p = 1 \Rightarrow S = \sum_{k=1}^{\infty} \frac{\ln k}{k}$

   (a) Integral Test

   (b) Comparison Test

   (a)

$$\int_1^{\infty} \frac{\ln x}{x} dx = \int_0^{\infty} u du = \infty$$

$$\left(u = \ln x \quad du = \frac{1}{x} dx \right)$$

   (b) when

$$k \geq 3, \ln k > 1$$

$$S' = \sum_{k=3}^{\infty} \frac{\ln k}{k} > \sum_{k=3}^{\infty} \frac{1}{k} \text{ diverges}$$

$$\Rightarrow S \text{ diverges by } p\text{-series theorem}$$

2)

$$p = \frac{3}{2} \Rightarrow S = \sum_{k=1}^{\infty} \frac{\ln k}{k^{\frac{3}{2}}}$$

   (a) Integral Test

$$\int_1^{\infty} \frac{\ln x}{x^{\frac{3}{2}}} dx \quad \left(\text{Let } \ln x = u\,,\ x^{-\frac{3}{2}} dx = dv\,,\ du = \frac{1}{x}\,,\ v = -2x^{-\frac{1}{2}} \right)$$

$$= \ln x \left(-2x^{-\frac{1}{2}} \right) \Big|_1^{\infty} - \int_1^{\infty} -2x^{-\frac{1}{2}} \frac{1}{x} dx$$

$$= 4 < \infty$$

(b)

$$\lim_{k\to\infty} \frac{\frac{\ln k}{k^{\frac{3}{2}}}}{\frac{1}{k^{\frac{5}{4}}}} \left(\text{or } k^{\frac{7}{6}}, k^{\frac{9}{8}} \ldots\right)$$
$$= \lim_{k\to\infty} \frac{\ln k}{k^{\frac{1}{4}}}$$
$$= \lim_{k\to\infty} \frac{\frac{1}{k}}{\frac{1}{4}k^{-\frac{3}{4}}}$$
$$= \lim_{k\to\infty} \frac{1}{\frac{1}{4}k^{\frac{1}{4}}}$$
$$= 0$$

$$\Rightarrow \text{By limit of comparison theorem, since } \left\langle \frac{1}{k^{5/4}} \right\rangle \text{ converges, } S \text{ converges.}$$

P.S

$$\lim_{k\to\infty} \frac{\frac{\ln k}{k^{\frac{3}{2}}}}{\frac{1}{k^{\frac{5}{4}}}} = 0 \leftarrow \text{constant}$$
$$\sum_{k=1}^{\infty} \frac{\ln k}{k^{\frac{3}{2}}} \quad \text{converges}$$

□

2. Determine whether the series converge or diverge.
   (a) $\sum_{k=1}^{\infty} \left(\sqrt{k} - \sqrt{k-1}\right)^{2k}$
   (b) $\sum_{k=1}^{\infty} \frac{(k!)^2}{(5k)!}$

   Sol.

   (a)

$$a_k = \left(\sqrt{k} - \sqrt{k-1}\right)^{2k}$$

$$\lim_{k\to\infty} (a_k)^{\frac{1}{k}}$$
$$= \lim_{k\to\infty} \left(\sqrt{k} - \sqrt{k-1}\right)^2$$
$$= \lim_{k\to\infty} \left(\frac{1}{\sqrt{k} + \sqrt{k-1}}\right)^2 = 0 < 1$$

$$\sum_{k=1}^{\infty} a_k \text{ converges by Root Test}$$

(b)

$$a_k = \frac{(k!)^2}{(5k)!}$$

$$\begin{aligned}
&\lim_{k\to\infty} \frac{a_{k+1}}{a_k} \\
=\ &\lim_{k\to\infty} \frac{(k+1)^2}{(5k+1)(5k+2)(5k+3)(5k+4)(5k+5)} \\
=\ &0 < 1 \\
\Rightarrow &\sum_{k=1}^{\infty} a_k \text{ converges by Ratio Test}
\end{aligned}$$

□

3. Find the interval of convergence of the series $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right) x^k$.

Sol.

$$\begin{aligned}
a_k &= \ln\left(\frac{k+1}{k}\right) \\
\lim_{k\to\infty} \frac{a_{k+1}}{a_k} &= \left.\frac{\ln\left(\frac{k+2}{k+1}\right)}{\ln\left(\frac{k+1}{k}\right)}\right|_{k\to\infty} \\
&= \left.\frac{\frac{1}{1+\frac{1}{k+1}} \cdot \frac{-1}{(k+1)^2}}{\frac{1}{1+\frac{1}{k}} \cdot \frac{-1}{k^2}}\right|_{k\to\infty} = 1
\end{aligned}$$

$$\left(\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| \le 1\,,\ |\ 1\cdot x\ | \le 1\,,\ |\ x\ | \le 1\,,\ \rho = 1\right)$$

check the endpoint

(i) $x = 1$

$$\sum_{k=1}^{\infty} \ln \frac{k+1}{k} x^k = \ln(k+1)\ |_{k\to\infty} = \infty$$

(ii) $x = -1$

$$\sum_{k=1}^{\infty} \ln \frac{k+1}{k} x^k = \sum_{k=1}^{\infty} \ln \frac{k+1}{k} (-1)^k < \infty$$

$$\left(1)\ \lim_{k\to\infty} \ln \frac{k+1}{k} = 0\,,\ 2)\ \frac{d}{dk}\left(\frac{k+1}{k}\right) = \frac{-1}{k^2} < 0\,,\ \text{Convergence Interval is } [-1, 1)\right)$$

□

4. (a) Find the Taylor series for $f(x) = (x^2 + x + 1)\sqrt{x+1}$ at $x = 0$ up to the third power of x.

(b) Let $f(x) = \ln\sqrt{\dfrac{1+x^2}{1-x^2}}$. Find $f^{(10)}(0)$.

Sol.

(a)

$$\begin{aligned}\sqrt{x+1} &= (1+x)^{\frac{1}{2}} \\ &= 1 + \frac{1}{2}x + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{1}{2!}x^2 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{3!}x^3 + \cdots \\ &\left((1+x)^m = 1 + C_1^m x + C_2^m x^2 + \cdots\right)\end{aligned}$$

$$\begin{aligned}\left(x^2+x+1\right)\sqrt{x+1} &= \left(x^2+x+1\right)\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots\right) \\ &= 1 + \frac{3}{2}x + \frac{11}{8}x^2 + \frac{7}{16}x^3 + \cdots\end{aligned}$$

(b)

$$\begin{aligned}f(x) &= \frac{1}{2}\left(\ln\left(1+x^2\right) - \ln\left(1-x^2\right)\right) \\ &= \frac{1}{2}\left(x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \frac{1}{4}x^8 + \frac{1}{5}x^{10} - \cdots\right) \\ &\quad - \frac{1}{2}\left(-x^2 - \frac{1}{2}x^4 - \frac{1}{3}x^6 - \frac{1}{4}x^8 - \frac{1}{5}x^{10} - \cdots\right) \\ &= x^2 + \frac{1}{3}x^6 + \frac{1}{5}x^{10} + \cdots\end{aligned}$$

$$f^{10}(0) = \frac{1}{5}\cdot 10!$$

P.S

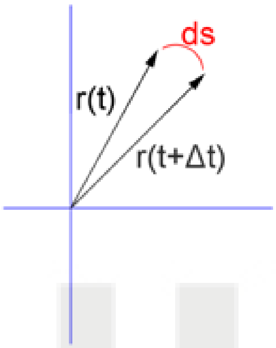
$$\begin{aligned}&\frac{d}{dx}\left(\ln\left(1+x^2\right)\right) \\ &= \frac{2x}{1+x^2} \\ &= 2x\left(1 - x^2 + x^4 - x^6 + \cdots\right) \\ &= 2x - 2x^3 + 2x^5 + \cdots\end{aligned}$$

$$\left(\ln\left(1+x^2\right) = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \cdots\right)$$

□

5. Find the curvature $\kappa(t)$ of the curve $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + \sqrt{2}t\mathbf{k}$.

Sol.



$$ds = r(t + \Delta t) - r(t)$$

$$\frac{ds}{dt} = r'(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

$$\mathbf{r}'(t) = \left(e^t, -e^{-t}, \sqrt{2}\right)$$

$$|\mathbf{r}'(t)| = \sqrt{e^{2t} + e^{-2t} + 2} = e^t + e^{-t}$$

$$\mathbf{T}(t) = \frac{1}{e^t + e^{-t}}\left(e^t, -e^{-t}, \sqrt{2}\right)$$

$$\mathbf{T}'(t) = \left(\frac{2}{(e^t + e^{-t})^2}, \frac{2}{(e^t + e^{-t})^2}, \frac{\sqrt{2}\,(e^{-t} - e^t)}{(e^t + e^{-t})^2}\right)$$

$$|\mathbf{T}'(t)| = \frac{\sqrt{2}}{(e^t + e^{-t})}$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}$$

□

6. Let $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

   (a) Find $\lim_{(x,y)\to(0,0)} f(x, y)$. Is f continuous at $(0, 0)$?

   (b) Find the partial derivative $\dfrac{\partial f}{\partial x}$ at $(x, y) = (0, 0)$ and at $(x, y) \neq (0, 0)$.

   (c) Is $\dfrac{\partial f}{\partial x}$ continuous at $(0, 0)$?

Sol.

(a)

$$f(x,y)=\frac{x^2y}{x^2+y^2}=\frac{r^3(\cos^2\theta\sin\theta)}{r^2}$$
$$=r\cos^2\theta\sin\theta$$
$$(x=r\cos\theta\,,\ y=r\sin\theta)$$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{r\to 0,\,\theta\text{ arbitrary}} r\cos^2\theta\sin\theta$$
$$=0=f(0,0)$$

$$\Rightarrow \text{continuous } at\ \ (0,0)$$

(b)

$$f_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=0$$
$$f_x(x,y)=\frac{\partial}{\partial x}\left(\frac{x^2y}{x^2+y^2}\right)=\frac{2xy^3}{(x^2+y^2)^2}\qquad ((x,y)\neq 0)$$

(c)

$$\lim_{(x,y)\to(0,0)} f_x(x,y)\overset{?}{=}f_x(0,0)$$

$$\because \lim_{y=x,\ x\to 0} f_x(x,y)=\frac{2x^4}{4x^4}=\frac{1}{2}\neq f_x(0,0)\Rightarrow f_x(x,y)\text{ is not continuous at }(0,0)$$

□

7. Let $u=u(x,y)$ be a function of rectangular coordinates x,y.Then u can be expressed in polar coordinates r , θ with $x=r\cos\theta$, $y=r\sin\theta$. Express $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in terms of r , θ , $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial \theta}$.

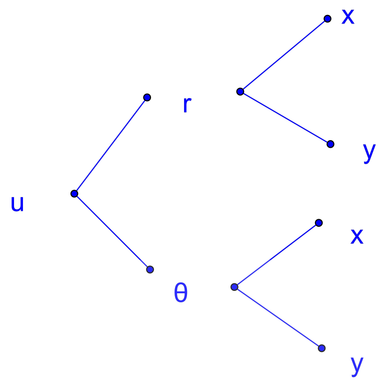
Sol.

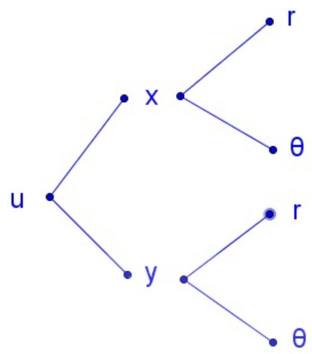
(1)

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$
$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$\frac{\partial r}{\partial x}=\frac{\partial\left(\sqrt{x^2+y^2}\right)}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}=\cos\theta,\qquad \frac{\partial r}{\partial y}=\sin\theta$$

$$\frac{\partial \theta}{\partial x}=\frac{\tan^{-1}\left(\frac{y}{x}\right)}{\partial x}=\frac{\frac{-y}{x^2}}{1+\left(\frac{y}{x}\right)^2}=\frac{-r\sin\theta}{r^2}=-\frac{\sin\theta}{r},\qquad \frac{\partial \theta}{\partial y}=\frac{\cos\theta}{r}$$





Therefore,

$$\frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$
$$\frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

(2)

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot \underbrace{\frac{\partial x}{\partial \theta}}_{\cos\theta} + \frac{\partial u}{\partial y}\cdot \underbrace{\frac{\partial y}{\partial \theta}}_{\sin\theta}$$
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot \underbrace{\frac{\partial x}{\partial \theta}}_{-r\sin\theta} + \frac{\partial u}{\partial y}\cdot \underbrace{\frac{\partial y}{\partial \theta}}_{r\cos\theta}$$

$$\begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{bmatrix}$$

□

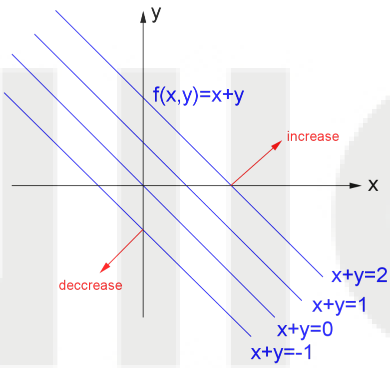
8. Let $f(x,y) = xe^y + \cos(xy)$.

(a) Find the direction (a unit vector $\mathbf{u}$) in which $f(x,y)$ increases most rapidly at $(2,0)$ (that is, $f'_{\mathbf{u}}(2,0)$ is maximal).

(b) Find the direction in which $f(x,y)$ decreases most rapidly at $(2,0)$.

(c) What are the directions of zero change in f at $(2,0)$?

Sol.



(a)

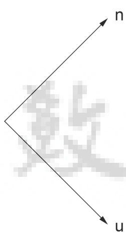
$$\nabla f(x,y) = \left(e^y - y\sin(xy),\, xe^y - x\sin(xy)\right)$$

$$\nabla f(2,0) = (1,2)$$

In the direction of $\frac{(1,2)}{\sqrt{5}}$, f increases most rapidly

(b) In the direction of $\frac{-(1,2)}{\sqrt{5}}$, f decreases most rapidly

(c) zero change $\to$ along the line



$$\vec{u} \cdot \frac{(1,2)}{\sqrt{5}} = 0$$

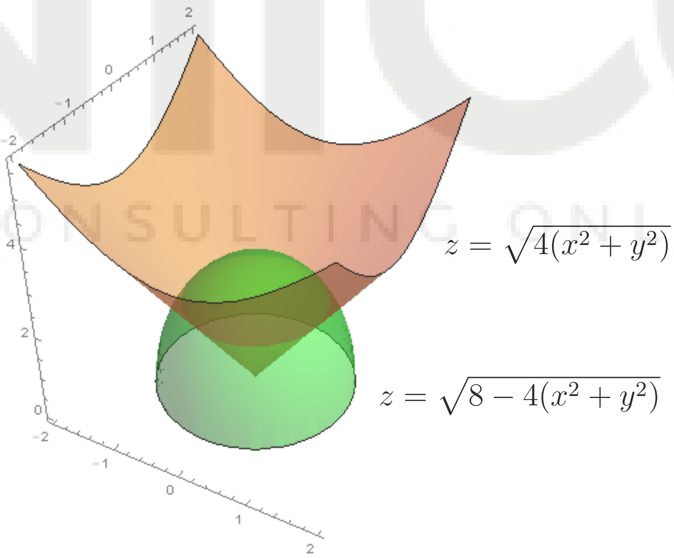
$$\vec{u} = \pm\frac{(2,-1)}{\sqrt{5}}$$

$\square$

101-2 微甲 07-11 班期末考試題及詳解

1. Find the volume of the solid bounded below by the cone $z^2 = 4(x^2 + y^2)$ and above by the ellipsoid $4(x^2 + y^2) + z^2 = 8$.

   Sol.



Solve for intersection of two surfaces:

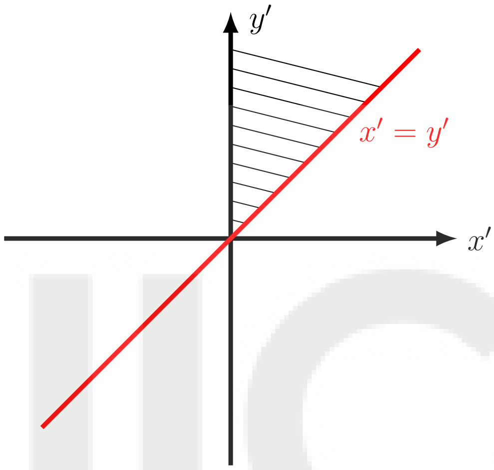
$$4(x^2 + y^2) + 4(x^2 + y^2) = 8$$
$$x^2 + y^2 = 1$$

$$\begin{aligned} V &= \iint_{x^2+y^2=1} (\underbrace{\sqrt{8 - 4(x^2 + y^2)} - \sqrt{4(x^2 + y^2)}}_{\text{height}}) dxdy \\ &= \int_0^{2\pi} \int_0^1 (2\sqrt{2 - r^2} - 2r) r dr d\theta \\ &= \frac{8\pi}{3}(\sqrt{2} - 1) \end{aligned}$$

□

2. Evaluate $\iint_\Omega e^{-4x^2-9y^2}dxdy$, where Ω is the region satisfy $2x \le 3y$ and $x \ge 0$.

Sol.



Let $x' = 2x$, $y' = 3y$

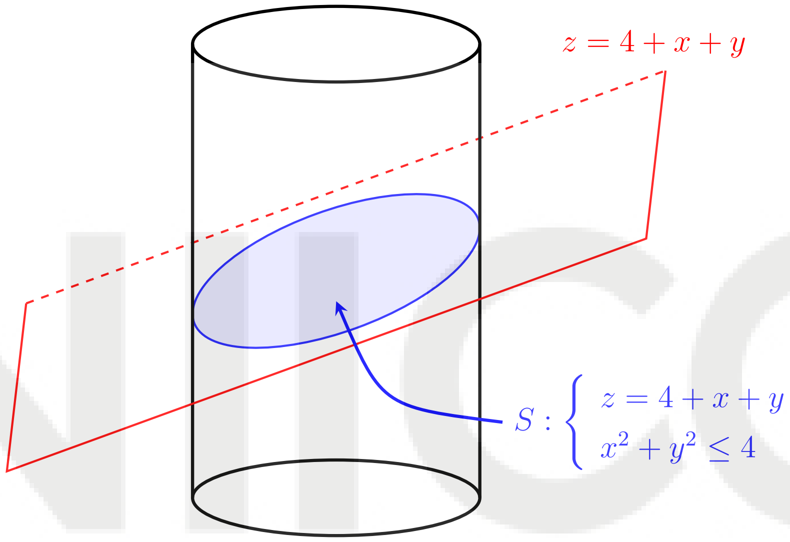
$$J = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{6} \qquad \Omega \to x' \le y'$$

$$\begin{aligned}
&\iiint_\Omega e^{-4x^2-9y^2}dxdy \\
=&\frac{1}{6}\iint_\Omega e^{-(x')^2-(y')^2}dx'dy' \\
=&\frac{1}{6}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta \\
=&\frac{1}{6}\frac{\pi}{4}\frac{-1}{2}e^{-r^2}\Big|_0^\infty \\
=&\frac{\pi}{48}
\end{aligned}$$

□

3. Evaluate the surface integral $\iint_S (x^2+y^2)z\,d\sigma$, where S is part of the plane $z = 4+x+y$ that lies inside the cylinder $x^2+y^2=4$.

Sol.



$$\mathbf{r} = (x, y, 4+x+y)$$

$$\Rightarrow \ S : \begin{cases} z = 4+x+y \\ x^2+y^2 \le 4 \end{cases}$$

$$d\sigma = \left(\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\right) dA$$

$$= \sqrt{3}dxdy \quad \text{(project } S \text{ onto } xy \text{ surface.)}$$

$$\iint_S (x^2+y^2)z\,d\sigma = \iint_{x^2+y^2\le 4} (x^2+y^2)(4+x+y)\sqrt{3}dxdy$$

$$= \int_0^{2\pi}\int_0^2 r^2(4 + r\cos\theta + r\sin\theta)\sqrt{3}r\,dr\,d\theta$$

$$= 32\sqrt{3}\pi$$

□

4. Find the line integral

$$\int_{\mathcal{C}} (2x\sin(\pi y) - e^z)dx + (\pi x^2\cos(\pi y) - 3e^z)dy - xe^z dz$$

along the curve $\mathcal{C} : \left\{(x, y, z) | z = \ln\sqrt{1 + x^2}, y = x, 0 \le x \le 1\right\}$

Sol.

Find the potential function

$$\begin{aligned}
&\frac{\partial f}{\partial x} = 2x\sin(\pi y) - e^z \\
&f(x, y, z) = x^2\sin(\pi y) - xe^z + g(y, z) \\
&\frac{\partial f}{\partial z} = -xe^z + \frac{\partial g}{\partial z} = -xe^z \\
&\therefore g = h(y) \\
\frac{\partial f}{\partial y} &= \pi x^2\cos(\pi y) + h'(y) \\
&= \pi x^2\cos(\pi y) - 3e^z \\
&\Rightarrow F = \nabla(x^2\sin(\pi y) - xe^z) - 3e^z\boldsymbol{j} \\
&\oint \boldsymbol{F}\cdot d\boldsymbol{r} = f(1, 1, \ln\sqrt{2}) - f(0, 0, 0) - 3\oint_c e^z dy \\
&= -\frac{5\sqrt{2} + 3\ln(\sqrt{2} + 1)}{2}
\end{aligned}$$

□

5. Evaluate $\oint_{r=1-\cos\theta} (x^2y + y)dx - (xy^2 - x)dy$ with the curve oriented counter-clockwise.

Sol.

Let $P = x^2y + y$, $Q = -xy^2 + x$, using Green's theorem,

$$\begin{aligned}
&\oint_r (x^2y + y)dx - (xy^2 - x)dy \\
&= \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \iint_\Omega \left(-x^2 - y^2\right) dxdy \\
&= \int_0^{2\pi}\int_0^{1-\cos\theta} -r^2 r dr d\theta \\
&= -\frac{1}{4}\int_0^{2\pi} (1 - \cos\theta)^4 d\theta = -\frac{35}{16}\pi
\end{aligned}$$

□

6. Let $\mathbf{V} = (2x - y)\mathbf{i} + (2y + z)\mathbf{j} + x^2y^2z^2\mathbf{k}$ and let S be the upper half of the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{25} = 1$. Find the flux of curl $\mathbf{V}$ in the direction of the upper unit normal $\mathbf{n}$ (pointing away from the origin).

Sol.

- Method 1 (Stoke's theorem)

$$\iint_S (\nabla \times \mathbf{V}) \cdot \mathbf{n} dS = \oint_c \mathbf{V} \cdot d\mathbf{r} \qquad \begin{cases} \mathbf{r} &= (2\cos\theta, 3\sin\theta, 0) \\ d\mathbf{r} &= (-2\sin\theta, 3\cos\theta, 0) \end{cases}$$

$$= \int_0^{2\pi} [(4\cos\theta - 3\sin\theta)(-2\sin\theta) + (6\sin\theta)(3\cos\theta)]d\theta$$

$$= \int_0^{2\pi} 10\sin\theta\cos\theta - 6\sin^2\theta d\theta = 6\pi$$

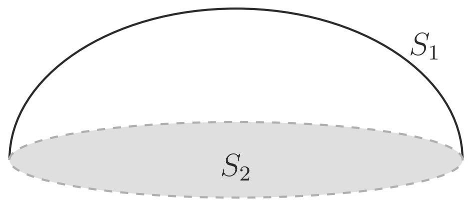
- Method 2 (divergence theorem)

$$\iint_{S_1+S_2} (\nabla \times \mathbf{V}) \cdot \mathbf{n}\, dS = \iiint \nabla \cdot (\nabla \times \mathbf{V}) d\mathbf{V} = 0$$

$$\iint_{S_1} (\nabla \times \mathbf{V}) \cdot \mathbf{n}\, dS_1 = -\iint_{S_2} (\nabla \times \mathbf{V}) \cdot \underline{\mathbf{n}}_{=(0,0,-1)} dS_2$$

$$= -\iint_{S_2} -1 \cdot dS_2$$

$$= \pi \cdot 2 \cdot 3 = 6\pi$$



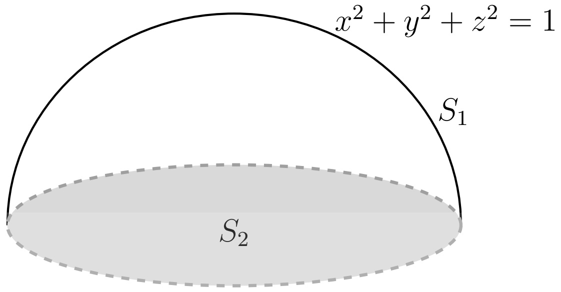
□

7. Evaluate the flux of

$$\mathbf{V}(x, y, z) = (xz^2 + y^2z)\mathbf{i} + \left(\frac{y^3}{3} + z\tan x\right)\mathbf{j} + (x^2z + 2y^2 + 1)\mathbf{k}$$

across S: the upper half sphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ with normal pointing away from the origin.

Sol.



$$S_1:\begin{cases}x^2+y^2+z^2=1\\ z\geq 0\end{cases}\quad S_2:\begin{cases}x^2+y^2\leq 1\\ z=0\end{cases}\quad \Omega:\begin{cases}x^2+y^2+z^2\leq 1\\ z\geq 0\end{cases}$$

$$\begin{aligned}\iiint_\Omega(\nabla\cdot\mathbf{V})d\Omega &= \iint_{S_1}\mathbf{V}\cdot d\mathbf{S}_1+\iint_{S_2}\mathbf{V}\cdot d\mathbf{S}_2\\ &=\iiint_\Omega(z^2+y^2+x^2)dxdydz\\ &=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1\rho^2\rho^2\sin\phi d\rho d\phi d\theta=\frac{1}{5}\Big|_0^1\cdot-\cos\phi\Big|_0^{\frac{\pi}{2}}\cdot\theta\Big|_0^{2\pi}\\ \iint_{S_2}\mathbf{V}\cdot d\mathbf{S}_2&=\iint_{S_2}\mathbf{V}\cdot(0,0,-1)dS_2=\iint-(2y^2+1)dS_2\\ &=\int_0^{2\pi}\int_0^1-[2(r\sin\theta)^2+1]rdrd\theta=-\frac{3\pi}{2}\\ &\therefore\iint_{S_1}\mathbf{V}\cdot d\mathbf{S}_1=\frac{2\pi}{5}-\frac{-3\pi}{2}=\frac{19\pi}{10}\end{aligned}$$

□

8. Find stationary points of $f(x,y)=3xy-x^3-y^3+2$, Determine which are local maximum, local minimum or a saddle point.

Sol.

$$\begin{aligned}\nabla f&=(3y-3x^2,3x-3y^2)=(0,0)\\ &\Rightarrow(x,y)=(0,0)\text{ and }(1,1)\\ D&=f_{xx}f_{yy}-f_{xy}^2=(-6x)(-6y)-3^2\\ &=36xy-9\\ D(0,0)&=-9<0\Rightarrow\text{saddle point}\\ D(1,1)&=27>0,\ f_{xx}(1,1)=-6<0,\ (1,1)\text{ is local max.}\end{aligned}$$

□

9. Use Lagrange multipliers to find the maximum and minimum of $f(x, y) = 3x^2 - 2y^2$ for x, y on the curve $2x^2 - 2xy + y^2 = 1$.

Sol.

$$\nabla(3x^2 - 2y^2) = \lambda\nabla(2x^2 - 2xy + y^2 - 1)$$

$$\Rightarrow \begin{cases} 6x = \lambda(4x - 2y) \\ -4y = \lambda(-2x + 2y) \\ 2x^2 - 2xy + y^2 = 1 \end{cases}$$

If $\lambda \neq 0$

$$\begin{cases} \dfrac{6x}{\lambda} - 4x = -2y \\ \dfrac{-4y}{\lambda} - 2y = -2x \end{cases} \Rightarrow \begin{cases} \dfrac{6}{\lambda} - 4 = -2\dfrac{y}{x} \\ \dfrac{-4}{\lambda} - 2 = -2\dfrac{x}{y} \end{cases}$$

$$\lambda = 2 \text{ or } \lambda = 3$$

$$\lambda = 2 \;\Rightarrow\; x = 2y$$

$$3x^2 - 2y^2 = 2 \text{ (max)}$$

$$\lambda = -3 \;\Rightarrow\; 3x = y$$

$$3x^2 - 2y^2 = -3 \text{ (min)}$$

If $\lambda = 0$

$$\lambda = 0 \Rightarrow (x, y) = (0, 0)$$

$$2x^2 - 2xy + y^2 = 0 \neq 1 \qquad (\rightarrow\leftarrow)$$

□