102 微甲 07-11 班試題及詳解

102-1 微甲 07-11 班期中考試題及詳解

1. Find $\dfrac{d}{dx}(\sec x)^x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

   Sol.

$$
\begin{aligned}
&\frac{d}{dx}(\sec x)^x \\
=& \frac{d}{dx}\left(e^{x\ln(\sec x)}\right) \\
=& e^{x\ln(\sec x)}\left(x\ln(\sec x)\right)' \\
=& e^{x\ln(\sec x)}\left(\ln(\sec x) + x\cdot\frac{1}{\sec x}\cdot\sec x\tan x\right) \\
=& (\sec x)^x(\ln(\sec x) + x\tan x)
\end{aligned}
$$

□

2. Evaluate $\displaystyle\int_0^1 \frac{(2+x)^2}{1+x^2}dx$.

   Sol.

$$
\begin{aligned}
&\int_0^1 \frac{(2+x)^2}{1+x^2}dx \\
=& \int_0^1 \frac{x^2+4x+4}{1+x^2}dx \\
=& \int_0^1 1 + \frac{4x}{1+x^2} + \frac{3}{1+x^2}dx \\
=& \int_0^1 1dx + 4\int_0^1 \frac{x}{1+x^2}dx + 3\int_0^1 \frac{1}{1+x^2}dx \\
=& x\Big|_0^1 + 4\ln(1+x^2)\cdot\frac{1}{2}\Big|_0^1 + 3\tan^{-1}x\Big|_0^1 \\
=& 1 + 2\ln 2 + \frac{3}{4}\pi
\end{aligned}
$$

□

3. (a) Find $\lim_{t\to 0^+} \frac{t-\ln(1+t)}{t^2}$

(b) Use (a) to find $\lim_{t\to 0^+} \frac{\sqrt{t-\ln(1+t)}}{t}$

Sol.

(a)

$$
\begin{aligned}
&\lim_{t\to 0^+} \frac{t-\ln(1+t)}{t^2} \left(\frac{0}{0}, \text{ by L'Hôpital's Rule}\right) \\
=& \lim_{t\to 0^+} \frac{1-\frac{1}{1+t}}{2t} \\
=& \left.\frac{t}{2t(1+t)}\right|_{t\to 0^+} \\
=& \frac{1}{2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
f(t) &= \frac{t-\ln(1+t)}{t^2} \\
g(t) &= \sqrt{t}
\end{aligned}
$$

Since $g(t)$ is continuous at $\frac{1}{2}$

$$\therefore \lim_{t\to 0^+} g(f(t)) = g\left(\lim_{t\to 0^+} f(t)\right) = \sqrt{\frac{1}{2}}$$

□

4. Evaluate $\lim_{x\to\infty} (2^x + 3^x + 5^x)^{\frac{1}{x}}$

Sol.

- Method 1

$$
\begin{aligned}
&\lim_{x\to\infty} (2^x + 3^x + 5^x)^{\frac{1}{x}} \\
=& \lim_{x\to\infty} e^{\frac{1}{x}\ln(2^x+3^x+5^x)} \\
=& \exp\left(\lim_{x\to\infty} \frac{\ln(2^x+3^x+5^x)}{x}\right) \quad \left(\frac{\infty}{\infty}, \text{ by L'Hôpital's Rule}\right) \\
=& \exp\left(\lim_{x\to\infty} \frac{2^x\ln 2 + 3^x\ln 3 + 5^x\ln 5}{2^x+3^x+5^x}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\left(\lim_{x\to\infty} \frac{5^x\left(\ln 5 + \left(\frac{2}{5}\right)^x \ln 2 + \left(\frac{3}{5}\right)^x \ln 3\right)}{5^x(1+(\frac{2}{5})^x+(\frac{3}{5})^x)}\right) \\
&= 5
\end{aligned}
$$

- Method 2

$$
\begin{aligned}
5^x < \quad & 2^x+3^x+5^x \quad < 3\cdot 5^x \\
\lim_{x\to\infty}(5^x)^{\frac{1}{x}} \le & \lim_{x\to\infty}(2^x+3^x+5^x)^{\frac{1}{x}} \le \lim_{x\to\infty}(3\cdot 5^x)^{\frac{1}{x}} \\
5 \le & \lim_{x\to\infty}(2^x+3^x+5^x)^{\frac{1}{x}} \le 3^0\cdot 5 = 5
\end{aligned}
$$

$$
\lim_{x\to\infty}(2^x+3^x+5^x)^{\frac{1}{x}} = 5 \quad \text{(by Squeeze Theorem)}
$$

□

5. Find the linear approximation of the function

$$
g(x) = \sin^{-1}\left(\frac{x-1}{x+1}\right) - \tan^{-1}(\sqrt{x}) \quad \text{at the point } x = 3
$$

Sol.

$$
\begin{aligned}
f(x+\Delta x) &\approx f(x) + \Delta x\cdot f'(x) \\
g(x) &\approx g(3) + g'(3)(x-3) \\
g'(x) &= \frac{\left(\frac{x-1}{x+1}\right)'}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} - \frac{1}{1+(\sqrt{x})^2}(\sqrt{x})' \\
&= \frac{\frac{2}{(x+1)^2}}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} - \frac{1}{1+x}\cdot\frac{1}{2}\frac{1}{\sqrt{x}} \\
g'(3) &= \frac{\frac{1}{8}}{\frac{\sqrt{3}}{2}} - \frac{1}{8\sqrt{3}} = \frac{1}{8\sqrt{3}} \\
g(3) &= \sin^{-1}\left(\frac{1}{2}\right) - \tan^{-1}(\sqrt{3}) = \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6} \\
\therefore g(x) &\approx g(3) + g'(3)(x-3) \\
&= -\frac{\pi}{6} + \frac{1}{8\sqrt{3}}(x-3)
\end{aligned}
$$

□

6. Let $y = f(x)$ satisfy $x^3 + 2xy + y^3 = 13$. Find y' and y'' at the point $x = 1,\ y = 2$.

Sol.

Differentiate with respect to x

$$\begin{aligned} & x^3 + 2xy + y^3 = 13 \\ \Rightarrow\ & 3x^2 + 2y + 2xy' + 3y^2y' = 0 \\ & y' = -\frac{3x^2 + 2y}{2x + 3y^2} \end{aligned}$$

Differentiate with respect to x again :

$$\begin{aligned} & 6x + 2y' + 2y' + 2xy'' + 6y(y')^2 + 3y^2y'' = 0 \\ & y'' = -\frac{6x + 4y' + 6y(y')^2}{2x + 3y^2} \end{aligned}$$

at $(1, 2)$:

$$\begin{aligned} y' &= -\frac{3 \cdot 1^2 + 2 \cdot 2}{2 \cdot 1 + 3 \cdot 2^2} = -\frac{7}{14} = -\frac{1}{2} \\ y'' &= -\frac{6 \cdot 1 + 4 \cdot (-\frac{1}{2}) + 6 \cdot 2 \cdot (-\frac{1}{2})^2}{2 \cdot 1 + 3 \cdot 2^2} = -\frac{7}{14} = -\frac{1}{2} \end{aligned}$$

□

7. Evaluate $\lim_{x \to 0^+} \dfrac{\int_0^{x^2} e^t \sqrt{t} \sin \sqrt{t} dt + x \cos x - x}{x^3}$

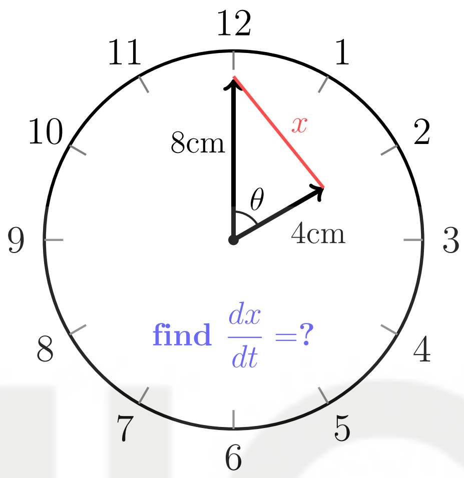
Sol.

$$\begin{aligned} & \lim_{x \to 0^+} \frac{\int_0^{x^2} e^t \sqrt{t} \sin \sqrt{t} dt + x \cos x - x}{x^3} \quad \left(\frac{0}{0}, \text{ by L'Hôpital's Rule}\right) \\ = & \lim_{x \to 0^+} \frac{e^{x^2} \sqrt{x^2} \sin \sqrt{x^2} \cdot (x^2)' + \cos x - x \sin x - 1}{3x^2} \\ = & \lim_{x \to 0^+} \frac{2e^{x^2} \sin x \cdot x^2 + \cos x - x \sin x - 1}{3x^2} \\ = & \lim_{x \to 0^+} \frac{2e^{x^2} \sin x \cdot x^2}{3x^2} + \lim_{x \to 0^+} \frac{\cos x - 1}{3x^2} + \lim_{x \to 0^+} \frac{-\sin x}{3x} \\ = & \lim_{x \to 0^+} \frac{2e^{x^2} \sin x}{3} + \lim_{x \to 0^+} \frac{-\cos x}{6} - \frac{1}{3} \lim_{x \to 0^+} \frac{\sin x}{x} \quad \text{(by L'Hôpital twice)} \\ = & \ 0 + \left(-\frac{1}{6}\right) + \left(-\frac{1}{3}\right) = -\frac{1}{2} \end{aligned}$$

□

8. The minute hand (分針) on a clock is 8cm long and the hour hand (時針) is 4cm long. How fast is the distance between the tips of the hands changing at two o'clock? Give your answer in the unit cm/hour.

Sol.



$$
\begin{aligned}
&x^2 = 8^2 + 4^2 - 2 \cdot 8 \cdot 4 \cos\theta \\
&2x\frac{dx}{dt} = 64 \sin\theta \frac{d\theta}{dt} \\
&\theta = 2\pi \cdot \frac{2}{12} = \frac{\pi}{3} \\
&\sin\theta = \frac{\sqrt{3}}{2} \\
&x = \sqrt{64 + 16 - 32} = 4\sqrt{3}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\theta}{dt} &= \frac{2\pi}{12} - \frac{2\pi}{1} = -\frac{11}{6}\pi \quad \text{(rad/hr)} \\
\frac{dx}{dt} &= \frac{1}{8\sqrt{3}} \cdot 64 \cdot \frac{\sqrt{3}}{2} \cdot -\frac{11}{6}\pi \\
&= 4 \cdot -\frac{11}{6}\pi \\
&= -\frac{22}{3}\pi \text{ (cm/hr)}
\end{aligned}
$$

□

9. Let $y = f(x) = x - \frac{x^2}{6} - \frac{2\ln x}{3}$, $x > 0$.

Answer the following question and give your reasons (including computations). Put "**None**" in the blank if the item asked does **not** exist.

(a) Find the interval(s) on which f is increasing.

Answer: ______

(b) Find the local maximal point(s) and minimal point(s) of f, if any.

Answer: $\begin{cases} \text{local maximal point(s) } (x,y) & = \text{______} \\ \text{local minimal point(s) } (x,y) & = \text{______} \end{cases}$

(c) Find the interval(s) on which f is concave up.

Answer: ______

(d) Find the inflection point(s) if any.

Answer: ______

(e) Sketch the graph of f. Indicate all information in (a)-(d).

Sol.

(a)

$$y' = 1 - \frac{x}{3} - \frac{2}{3x}$$

$$y' > 0 \Rightarrow x^2 - 3x + 2 < 0 \quad (\because x > 0)$$

$$\Rightarrow 1 < x < 2$$

(b)

$$\begin{cases} \text{local max} & = (2, \frac{4 - 2\ln 2}{3}) \\ \text{local min} & = (1, \frac{5}{6}) \end{cases}$$

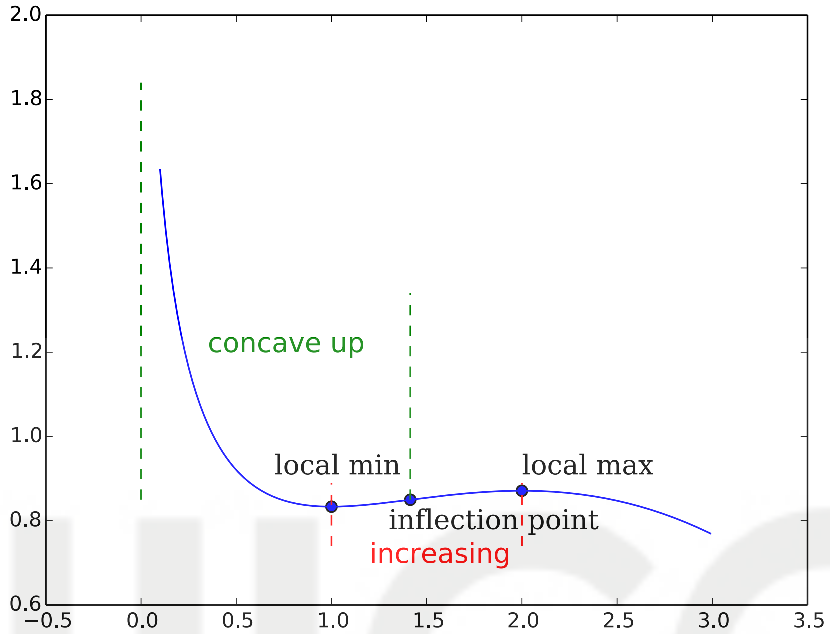
(c) concave up $\Rightarrow f''(x) > 0$

$$-\frac{1}{3} + \frac{2}{3x^2} > 0$$

$$\frac{2}{x^2} - 1 > 0 \Rightarrow 0 < x < \sqrt{2} \quad (\because x > 0)$$

(d) $f''(x) = 0 \Rightarrow x = \sqrt{2}, -\sqrt{2}(\to\leftarrow)$, inflection point $= (\sqrt{2}, \sqrt{2} - \frac{1 + \ln 2}{3})$

(e)


concave up
local min
local max
inflection point
increasing
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5

□

102-1 微甲 07-11 班期末考試題及詳解

1. Let $h(x) = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x}), 0 \leq x \leq 1$.

   (a) Find the length of the curve $y = h(x)$.

   (b) Find the area of the surface generated by rotating the curve $y = h(x)$ about the x-axis.

Sol.

(a)

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}\ dx$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{1}{2}(1-2x)}{\sqrt{x-x^2}} + \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} \\
&= \sqrt{\frac{1-x}{x}}, \quad 0 < x < 1 \\
\Rightarrow ds &= \sqrt{\frac{1}{x}} dx \\
s &= \int_0^1 \frac{1}{\sqrt{x}} dx \\
&= 2\sqrt{x}\Big|_0^1 \\
&= 2
\end{aligned}$$

(b)

$$\begin{aligned}
A &= \int 2\pi y ds \\
&= 2\pi \int_0^1 \left(\sqrt{x-x^2} + \sin^{-1}(\sqrt{x})\right) \frac{1}{\sqrt{x}} dx \\
&= 2\pi \int_0^1 \left(\sqrt{1-x} + \frac{\sin^{-1}(\sqrt{x})}{\sqrt{x}}\right) dx
\end{aligned}$$

$$= 2\pi\left[\frac{-2}{3}(1-x)^{\frac{3}{2}}\Big|_0^1 + 2\sqrt{x}\sin^{-1}(\sqrt{x})\Big|_0^1 - \int_0^1 \sqrt{x}\frac{\frac{1}{2\sqrt{x}}}{\sqrt{1-x}}dx\right]$$
$$= 2\pi(\pi - \frac{4}{3})$$

□

2. Solve $xy' - 3y = 5x^3$
   (a) with the initial condition $y(1) = 2$.
   (b) with the initial condition $y(-1) = 2$.

Sol.

$$xy' - 3y = 5x^3 \quad \Rightarrow \quad y' - \frac{3}{x}y = 5x^2$$

$$(Iy)' = I(y' - \frac{3}{x}y) = 5x^2 I$$
$$I'y + Iy' = Iy' - \frac{3}{x}Iy$$
$$\Rightarrow \frac{I'}{I} = -\frac{3}{x}$$
$$\Rightarrow \ln I = -3\ln|x| + C, \text{ let } C = 0$$
$$\Rightarrow I = \frac{1}{x^3}$$
$$\therefore (\frac{1}{x^3}y)' = \frac{5}{x}$$
$$\Rightarrow \frac{1}{x^3}y = 5\ln|x| + C$$
$$\Rightarrow y = 5x^3\ln|x| + Cx^3$$

(a) $y(1) = 2,\ C = 2,$
$$\Rightarrow y = 5x^3\ln|x| + 2x^3$$

(b) $y(-1) = 2,\ C = -2,$
$$\Rightarrow y = 5x^3\ln|x| - 2x^3$$

□

3. Let $y = h(x)$ be decreasing on $[0, \frac{\pi}{2})$ and is continuously differentiable on $(0, \frac{\pi}{2})$ with $h(0) = 0$. Let $s(x)$ denote the arc length of $y = h(x)$ from $(0, 0)$ to $(x, h(x))$.

   (a) Write down the formula for $s(x)$.

   (b) Suppose that $s(x)$ is also given by $s(x) = \int_0^x e^{-h(t)} dt$. Find the function $h(x)$ explicitly.

   (c) Find the function $s(x)$ explicitly.

Sol.

(a)

$$s(x) = \int_0^x \sqrt{1 + (h'(t))^2} dt$$

$$\left(ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \right)$$

(b)

$$\begin{aligned}
&\int_0^x \sqrt{1 + (h'(t))^2} dt = \int_0^x e^{-h(t)} dt \\
&\Rightarrow \sqrt{1 + (h'(x))^2} = e^{-h(x)} \\
&\Rightarrow 1 + (h'(x))^2 = e^{-2h(x)} \\
&\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = e^{-2y} \\
&\Rightarrow \frac{dy}{dx} = -\sqrt{e^{-2y} - 1} \\
&\Rightarrow -\frac{1}{\sqrt{e^{-2y} - 1}} dy = dx
\end{aligned}$$

Let $e^{-y} = \sec\theta$, then

$$\begin{aligned}
-e^{-y} dy &= \sec\theta \tan\theta d\theta \\
-dy &= e^{y} \sec\theta \tan\theta d\theta \\
&= \cos\theta \sec\theta \tan\theta d\theta
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int \frac{\cos\theta \sec\theta \tan\theta}{\tan\theta} d\theta = \int dx \\
&\Rightarrow \theta + C = x
\end{aligned}$$

$$\Rightarrow x = \sec^{-1}(e^{-y}) + C$$

$$\because (x, y) = (0, 0) \quad \Rightarrow \; C = 0$$

$$\Rightarrow e^{-y} = \sec x$$

$$\Rightarrow h(x) = y = -\ln|\sec x| = \ln|\cos x|$$

(c)

$$\begin{aligned} s(x) &= \int_0^x \sqrt{1 + \left(\frac{\sin t}{\cos t}\right)^2}\, dt \\ &= \int_0^x \sec t dt \\ &= \ln|\sec x + \tan x| \end{aligned}$$

□

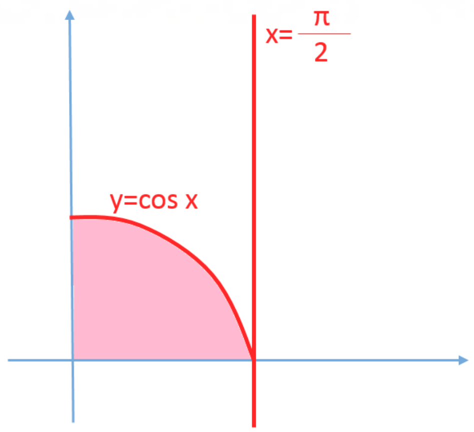
4. Let Ω be the region bounded by $y = \cos x, y = 0, x = 0$ and $x = \dfrac{\pi}{2}$.

   (a) Find the volume of the solid obtained by revolving Ω about x-axis.

   (b) Find the volume of the solid obtained by revolving Ω about y-axis.

   (c) Find the centroid of Ω.

Sol.



(a)

$$\begin{aligned} V &= \int_0^{\pi/2} \pi y^2 dx = \pi \int_0^{\pi/2} \cos^2(x) dx \\ &= \pi \int_0^{\pi/2} \frac{\cos 2x + 1}{2} dx \end{aligned}$$

$$
\begin{aligned}
&= \pi \left(\frac{\sin 2x}{4} + \frac{x}{2} \right) \Bigg|_0^{\pi/2} \\
&= \frac{\pi^2}{4}
\end{aligned}
$$

(b)

$$
\begin{aligned}
V &= \int \pi x^2 dy = \int \pi x^2 (-\sin x) dx \\
&= -\pi \int \underbrace{x^2}_{A} \underbrace{\sin x dx}_{dB} \\
&= -\pi \left[x^2(-\cos x) \Big|_{\frac{\pi}{2}}^{0} - \int (-\cos x) 2x dx \right] \\
&= -\pi \left(2 \int \underbrace{x}_{A} \underbrace{\cos x dx}_{dB} \right) \\
&= -2\pi \left(x \sin x \Big|_{\frac{\pi}{2}}^{0} - \int_{\frac{\pi}{2}}^{0} \sin x dx \right) \\
&= \pi^2 - 2\pi
\end{aligned}
$$

(c)

$$
\begin{aligned}
\bar{x} &= \frac{\int_0^{\pi/2} x \cos x dx}{\int_0^{\pi/2} \cos x dx} = \frac{\pi}{2} - 1 \\
\bar{y} &= \frac{\int_0^{\pi/2} \frac{1}{2} \cos^2 x dx}{\int_0^{\pi/2} \cos x dx} = \frac{\pi}{8}
\end{aligned}
$$

or use Pappus Theorem:

$$2\pi A\bar{x} = V (\text{resolve about } y\text{-axis})$$

$$2\pi A\bar{y} = V (\text{resolve about } x\text{-axis})$$

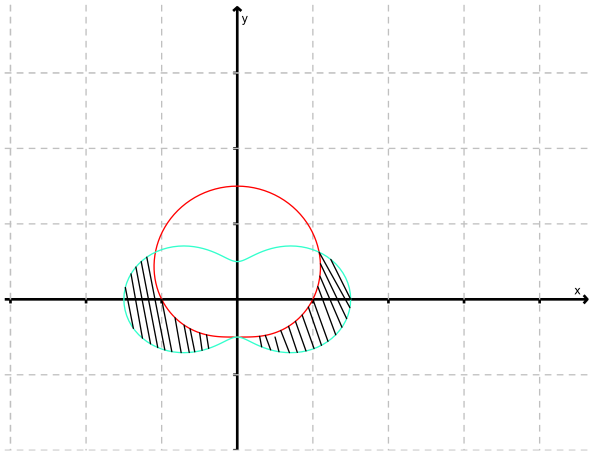
□

5. Find the area of the region that lies inside the curve $r = 2 + \cos 2\theta$ but outside the curve $r = 2 + \sin\theta$.

Sol.

(i) Find intersections

$$2 + \sin\theta = 2 + \cos 2\theta$$



$$\begin{aligned}
&\sin\theta = 1 - 2\sin^2\theta \\
\Rightarrow\ &\sin\theta = -1 \text{ or } \frac{1}{2} \\
\Rightarrow\ &\theta = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}
\end{aligned}$$

(ii)

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} [(2+\cos 2\theta)^2 - (2+\sin\theta)^2]d\theta \times 2 \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (4\cos 2\theta + \cos^2 2\theta - 4\sin\theta - \sin^2\theta)d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (4\cos 2\theta + \frac{1+\cos 4\theta}{2} - 4\sin\theta - \frac{1-\cos 2\theta}{2})d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\frac{9}{2}\cos 2\theta + \frac{1}{2}\cos 4\theta - 4\sin\theta\right) d\theta \\
&= \frac{9}{4}\sin 2\theta + \frac{1}{8}\sin 4\theta + 4\cos\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{6}} \\
&= \frac{51}{16}\sqrt{3}
\end{aligned}$$

□

6. Find the arc length of the curve $x = t\sin 2t, y = t\cos 2t, 0 \le t \le 1$.

Sol.

$$\begin{aligned}
dl &= \sqrt{(dx)^2 + (dy)^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt \\
x'(t) &= (t\sin 2t)' = \sin(2t) + 2t\cos(2t)
\end{aligned}$$

$$y'(t) = (t\cos 2t)' = \cos(2t) - 2t\sin(2t)$$

$$L = \int_0^1 \sqrt{(x')^2 + (y')^2}dt = \int_0^1 \sqrt{1+4t^2}dt$$

Let $2t = \tan\theta \ \Rightarrow dt = \frac{1}{2}\sec^2\theta d\theta$

$$\begin{aligned} L &= \frac{1}{2}\int_0^{\tan^{-1}(2)} |\sec\theta|\sec^2\theta d\theta \\ &= \frac{1}{2}\int_0^{\tan^{-1}(2)} \sec^3\theta d\theta \\ &= \frac{1}{2}\cdot\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)\Big|_0^{\tan^{-1}(2)} \\ &= \frac{1}{4}[2\sqrt{5} + \ln(2+\sqrt{5})] \end{aligned}$$

Note that

$$\begin{aligned} \int \sec^3\theta d\theta &= \int \sec\theta\sec^2\theta d\theta \\ &= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)d\theta \\ &= \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| - \int \sec^3\theta d\theta \end{aligned}$$

$$\Rightarrow \int \sec^3\theta = \frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|)$$

□

7. Let $P(x) = x^3 + x^2 + x + 1$ and $Q(x) = x^3 - x^2 + x + 1$. Evaluate $\int \frac{Q(x)}{P(x)}dx$. Note that $P(-1) = 0$.

Sol.

$$\begin{aligned} &\int \frac{x^3 - x^2 + x + 1}{x^3 + x^2 + x + 1}dx \\ =& \int 1 - \frac{2x^2}{x^3+x^2+x+1}dx \\ =& \int 1 - \frac{2x^2}{(x^2+1)(x+1)}dx \\ =& \int 1 - \frac{1}{x+1} + \frac{1-x}{x^2+1}dx \\ =& \ x - \ln|x+1| + \tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C \end{aligned}$$

□

8. Determine the values of $\alpha > 0$ such that $\int_1^\infty \frac{\ln x}{x^\alpha}dx$ is convergent.

Sol.

(a) If $\alpha \neq 1$

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^\alpha}dx &= \ln x \frac{x^{1-\alpha}}{1-\alpha}\Big|_1^\infty - \frac{1}{1-\alpha}\int_1^\infty x^{-\alpha}dx \\ &= \left[\frac{x^{1-\alpha}}{1-\alpha}\ln x - \frac{x^{1-\alpha}}{(1-\alpha)^2}\right]\Big|_{x\to\infty} + \frac{1}{(1-\alpha)^2}\end{aligned}$$

(i) If $0 < \alpha < 1$

$$\lim_{x\to\infty} \frac{(1-\alpha)\ln x - 1}{(1-\alpha)^2 x^{\alpha-1}} = \infty$$

(ii) If $\alpha > 1$, by L'Hospital Rule,

$$\begin{aligned}&\lim_{x\to\infty} \frac{(1-\alpha)\ln x - 1}{(1-\alpha)^2 x^{\alpha-1}} \\ &= \frac{\frac{1}{x}(1-\alpha)}{(1-\alpha)^2(\alpha-1)x^{\alpha-2}}\Big|_{x\to\infty} \\ &= \frac{-1}{(1-\alpha)^2 x^{\alpha-1}}\Big|_{x\to\infty} \\ &= 0\end{aligned}$$

(b) If $\alpha = 1$

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^\alpha}dx &= \int_1^\infty \frac{\ln x}{x}dx \quad (\text{Let } u = \ln x,\ du = \frac{1}{x}dx) \\ &= \frac{1}{2}(\ln x)^2\Big|_{x\to\infty} \\ &= \infty\end{aligned}$$

$$\therefore \int_1^\infty \frac{\ln x}{x^\alpha}dx \text{ is convergent for } \alpha > 1$$

□

102-2 微甲 07-11 班期中考試題及詳解

1. Find the values of ρ for the convergence of the series below

   (a) $\displaystyle\sum_{n=0}^{\infty} e^{n(\rho^2-\rho-2)}$,

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}-1}{n^{\rho}}$.

Sol.

(a)

$$\begin{aligned} e^{\rho^2-\rho-2} < 1 &\Rightarrow \rho^2-\rho-2<0 \\ &\Rightarrow \ -1<\rho<2 \end{aligned}$$

(b)

$$e^{\frac{1}{n}}-1 = \left(1+\frac{1}{n}+\frac{\left(\frac{1}{n}\right)^2}{2!}+\cdots\right)-1 \approx \frac{1}{n}$$

$$\lim_{n\to\infty}\frac{\frac{e^{\frac{1}{n}}-1}{n^{\rho}}}{\frac{1}{n^{\rho+1}}} = \lim_{n\to\infty}\frac{e^{\frac{1}{n}}-1}{\frac{1}{n}} = \lim_{t\to 0}\frac{e^t-1}{t} = \lim_{t\to 0} e^t = 1$$

$\therefore$ By limit comparision test, $\langle\frac{e^{\frac{1}{n}-1}}{n^{\rho}}\rangle$ and $\langle\frac{1}{n^{\rho+1}}\rangle$ both converge or both diverge

$\therefore$ $\rho+1>1 \ \Rightarrow \ \rho>0$

□

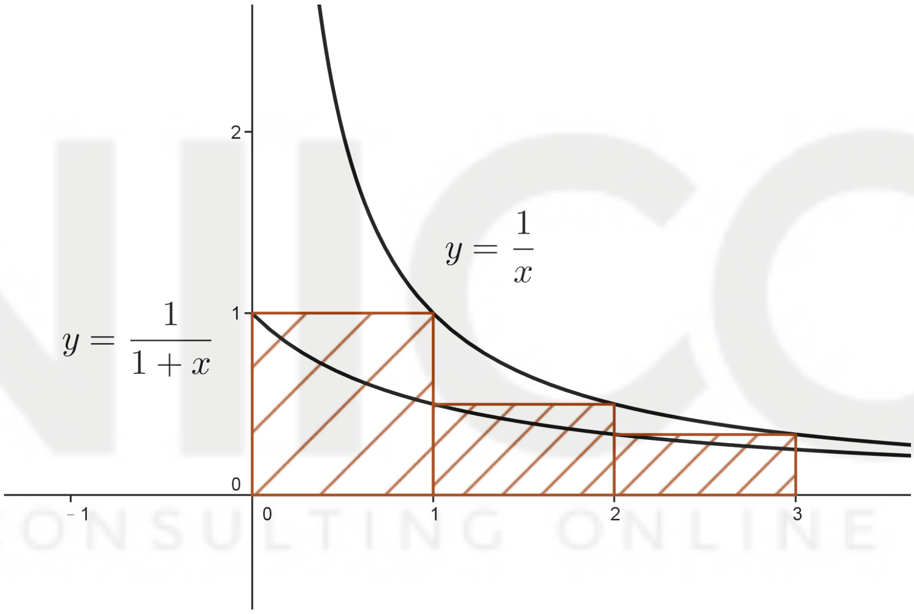
2. (a) Prove $\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln n$.

(b) Test for convergence of $\sum_{n=1}^{\infty} a_n$, where $a_n = \dfrac{1}{1 + \frac{1}{2} + \cdots + \frac{1}{n}}$.

Sol.

(a)

$$\int_0^n \frac{1}{1+x} dx < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{1}{x}$$
$$\ln(n+1) < 1 + \frac{1}{2} + \cdots + \frac{1}{n} < 1 + \ln(n)$$



(b)

$$a_n = \frac{1}{1 + \frac{1}{2} + \cdots + \frac{1}{n}} > \frac{1}{1 + \ln(n)} > \frac{1}{1+n}$$
$$\lim_{n\to\infty} \frac{\frac{1}{1+n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{1+n} = 1, \text{ and } \sum \frac{1}{n} \text{ diverges}$$

by comparison test

$$\sum \frac{1}{1+n} \text{ diveges } \Rightarrow \sum a_n \text{ diverges}$$

or

$$\because \int_1^\infty \frac{1}{1+x} dx = \infty \Rightarrow \text{by integral test, } \sum \frac{1}{1+n} \text{ diverges} \Rightarrow \sum a_n \text{ diverges}$$

□

3. Let $f(x,y) = \sin(x-y)e^{-x^2-y^2},\ P = (\sqrt{2}, \sqrt{2})$.

(a) Find the maximum rate of change of f at P.

(b) Find the direction in which the maximum rate of change occurs.

(c) Find the directional derivative $D_{\mathbf{u}}(P)$, where $\mathbf{u} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Sol.

(a) (b)

$$\begin{aligned}\nabla f(x,y) &= \Big(\cos(x-y)e^{-x^2-y^2} - 2x\sin(x-y)e^{-x^2-y^2}, \\ &\quad -\cos(x-y)e^{-x^2-y^2} - 2y\sin(x-y)e^{-x^2-y^2}\Big)\\ \nabla f(\sqrt{2},\sqrt{2}) &= (e^{-4}, -e^{-4})\\ |\nabla f(\sqrt{2},\sqrt{2})| &= \sqrt{2}e^{-4}\\ \text{direction } &= (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\end{aligned}$$

(c)

$$\begin{aligned}D_{\mathbf{u}}(P) &= \nabla f(P)\cdot\mathbf{u}\\ &= (e^{-4}, -e^{-4})\cdot(\frac{1}{2}, \frac{\sqrt{3}}{2})\\ &= \frac{1-\sqrt{3}}{2e^4}\end{aligned}$$

□

4. Let $\mathbf{r}(t) = \left\langle \frac{t^4}{2}, t, \frac{4}{5}t^{\frac{5}{2}} \right\rangle$ for $t \geq 0$.

(a) Find the length of the arc $0 \leq t \leq 2$ of $\mathbf{r}(t)$.

(b) Find the curvature $\kappa(t)$.

(c) Find $\mathbf{T}(1)$, $\mathbf{N}(1)$ and $\mathbf{B}(1)$, the principal unit normal vector and the binormal unit vector when $t = 1$ respectively.

Sol.

(a)

$$\mathbf{r}(t) = \left(\frac{t^4}{2}, t, \frac{4}{5}t^{\frac{5}{2}}\right)$$

$$\mathbf{r}'(t) = (2t^3, 1, 2t^{\frac{3}{2}})$$
$$|\mathbf{r}'(t)| = 1 + 2t^3$$
$$L = \int_0^2 (1 + 2t^3)dt = 10$$

(b)

$$\mathbf{r}''(t) = (6t^2, 0, 3t^{\frac{1}{2}})$$
$$\mathbf{r}'(t) \times \mathbf{r}''(t) = (3t^{\frac{1}{2}}, 6t^{\frac{7}{2}}, -6t^2)$$
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{3\sqrt{t}}{(2t^3+1)^2}$$

(c)

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{1+2t^3}(2t^3, 1, 2t^{\frac{3}{2}})$$
$$\mathbf{T}(1) = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$$
$$\mathbf{T}'(t) = \frac{1}{(1+2t^3)^2}(6t^2, -6t^2, -6t^{\frac{7}{2}} + 3t^{\frac{1}{2}})$$
$$\mathbf{T}'(1) = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right), \ \mathbf{N}(1) = \frac{\mathbf{T}'(1)}{|\mathbf{T}'(1)|} = \left(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$$
$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

□

5. Find the extreme values of $f(x,y) = x^2y - xy + xy^2$ on $x^2 + xy + y^2 - x - y = 1$.

Sol.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \Rightarrow \begin{cases} y(2x+y-1) = \lambda(2x+y-1) \\ x(x+2y-1) = \lambda(x+2y-1) \\ x^2 + xy + y^2 - x - y = 1 \end{cases}$$

(a) $2x + y - 1 = 0 \ \Rightarrow \ (x,y) = \left(\frac{-1}{3}, \frac{5}{3}\right)$ or $(1,-1)$

(b) $2x + y - 1 \neq 0 \ \Rightarrow \ \lambda = y$

(i) $x + 2y - 1 = 0 \ \Rightarrow \ (x,y) = \left(\frac{5}{3}, \frac{-1}{3}\right)$ or $(1,-1)$

(ii) $x + 2y - 1 \neq 0 \ \Rightarrow \ \lambda = x = y \ \Rightarrow \ (x,y) = \left(\frac{-1}{3}, \frac{-1}{3}\right)$ or $(1,1)$

We find that $\begin{cases} f(1,-1) = f(-1,1) = f(1,1) = 1 & \boxed{\text{Max}} \\ f\left(\frac{-1}{3}, \frac{5}{3}\right) = f\left(\frac{5}{3}, \frac{-1}{3}\right) = f\left(\frac{-1}{3}, \frac{-1}{3}\right) = \frac{-5}{27} & \boxed{\text{Min}} \end{cases}$

□

6. Find the local maximum, and local minimum values and saddle point(s) of $f(x,y) = y^3 + 3x^2y - 3x^2 - 3y^2 + 3$.

Sol.

$$\begin{cases} \dfrac{\partial f}{\partial x} = 0 \;\Rightarrow\; 6xy - 6x = 0 \\ \dfrac{\partial f}{\partial y} = 0 \;\Rightarrow\; 3y^2 + 3x^2 - 6y = 0 \end{cases}$$

$$\Rightarrow \begin{matrix} \text{(a) } x = 0 \;\Rightarrow\; y = 0, 2 \\ \text{(b) } y = 1 \;\Rightarrow\; x = -1, 1 \end{matrix}$$

we have 4 critical points

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = (6y-6)^2 - (6x)^2 \qquad (f_{xy} = 6x)$$

$$\begin{aligned} &D(0,0) = 36 > 0 \quad f_{xx} = -6 < 0 && \Rightarrow \text{ max} \\ &D(0,2) = 36 > 0 \quad f_{xx} = 6 > 0 && \Rightarrow \text{ min} \\ &D(-1,1) = -36 < 0,\; D(1,1) = -36 < 0 && \Rightarrow \text{ saddle point} \end{aligned}$$

□

7. (a) Find the radius of convergence and the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}}$.

   (b) Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}}$ when the power series is convergent. Evaluate $f^{(3)}(1)$.

Sol.

(a)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n}}{\sqrt{n+1}}\right|\left|\frac{x-1}{-2}\right| = \left|\frac{x-1}{-2}\right| < 1$$

$$\Rightarrow |x-1| < 2$$

check end points

(i) $x = 3 \;\Rightarrow\; \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges $\left(\because \begin{matrix} ① \; a_{n+1} < a_n \quad ② \; \displaystyle\lim_{n\to\infty} a_n = 0 \\ ③ \text{‘} a_n \text{ is alternate series} \end{matrix}\right)$

(ii) $x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (by p-series theorem)

So the radius of convergence is $(-1, 1]$.

(b)

$$f(x) = \sum_{n=1}^{\infty} \frac{(x-1)^n}{(-2)^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$
$$n = 3 \to \frac{1}{(-2)^3\sqrt{3}} = \frac{f^{(3)}(1)}{3!}$$
$$\therefore\ f^{(3)}(1) = \frac{3!}{-8\sqrt{3}} = -\frac{\sqrt{3}}{4}$$

□

8. (a) Write down the general terms the MacLaurin series of $\sin x$ and $\sin^{-1} x$.
   (b) Find their radii of convergence.
   (c) Find $\lim_{x\to 0} \frac{\sin x \cdot \sin^{-1} x - x^2}{x^6}$.

Sol.

(a) (b)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n+1}$$
$$\sin^{-1}(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} (-t^2)^k dt$$
$$= \sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} (-1)^k \frac{1}{2k+1} x^{2k+1}$$

(c)

$$\lim_{x\to 0} \frac{\sin x \cdot \sin^{-1} x - x^2}{x^6}$$
$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots\right)\left(x + \frac{x^3}{6} + \frac{3}{40}x^5 \cdots\right) - x^2}{x^6}$$
$$= \frac{3}{40} - \frac{1}{36} + \frac{1}{120} = \frac{1}{18}$$

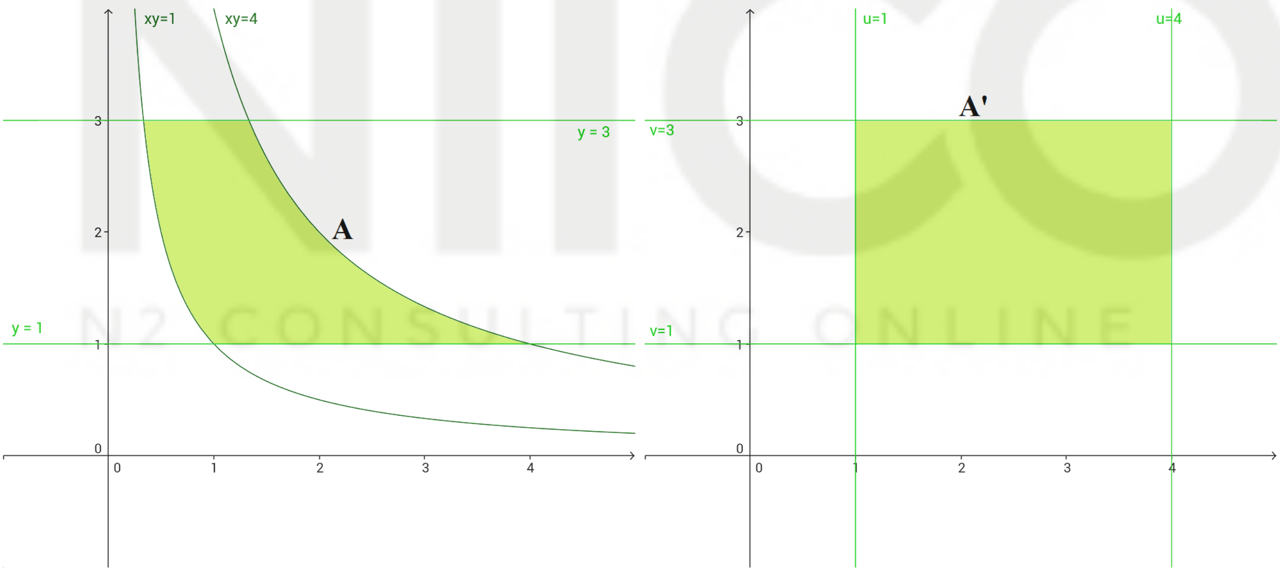
□

102-2 微甲 07-11 班期末考試題及詳解

1. Evaluate $\iint_A e^{xy}dxdy$, where A is the region enclosed by $xy = 1$, $xy = 4$, $y = 1$ and $y = 3$.

Sol.

Let $u = xy,\ v = y \quad \Rightarrow x = \dfrac{u}{y},\ y = v$



$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{y} & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{y} = \frac{1}{v}$$

$$\begin{aligned} &\iint_A e^{xy}dxdy \\ =&\iint_{A'} e^u \cdot \frac{1}{v}dudv \\ =&\int_1^3\int_1^4 e^u \cdot \frac{1}{v}dudv \end{aligned}$$

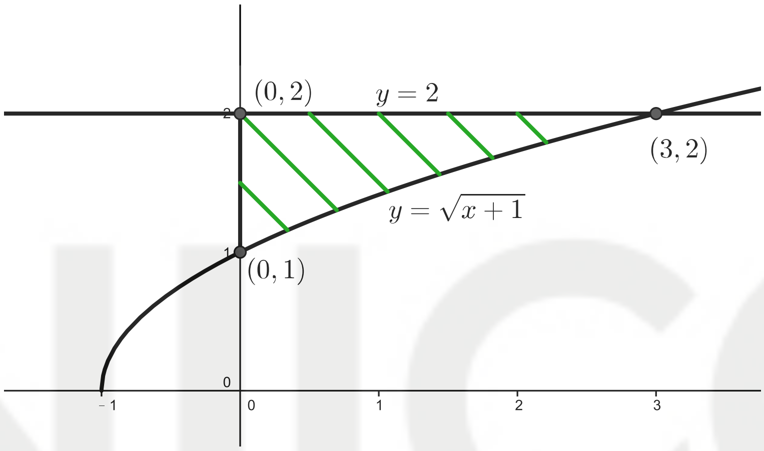
$$= (e^4 - e)\ln 3$$

□

2. Sketch the region of integration and evaluate the integral $\int_0^3 \int_{\sqrt{x+1}}^2 e^{\frac{x}{y+1}} dydx$.

Sol.

The region of integration is shown in the figure.



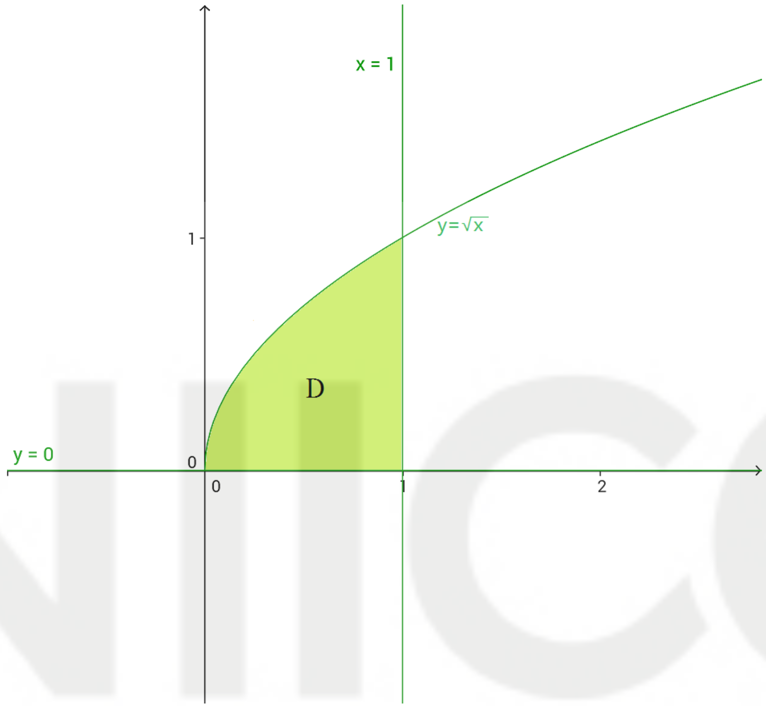
$$\begin{aligned}
&\int_0^3 \int_{\sqrt{x+1}}^2 e^{\frac{x}{y+1}} dydx \\
=& \int_1^2 \int_0^{y^2-1} e^{\frac{x}{y+1}} dxdy \quad (y = \sqrt{x+1} \Rightarrow x = y^2 - 1) \\
=& \int_1^2 (y+1)e^{\frac{x}{y+1}} \Big|_0^{y^2-1} dy \\
=& \int_1^2 (y+1)(e^{y-1} - 1)dy \\
=& \int_1^2 y \cdot e^{y-1} dy + \int_1^2 e^{y-1} dy - \int_1^2 (y+1)dy \\
=& \, ye^{y-1} \Big|_1^2 - \int_1^2 e^{y-1} dy + \int_1^2 e^{y-1} dy - (\frac{y^2}{2} + y)\Big|_1^2 \\
=& \, 2e - 1 - \frac{5}{2} = 2e - \frac{7}{2}
\end{aligned}$$

□

3. Let D be the bounded region in the first quadrant enclosed by $y = 0,\ x = 1$, and $y = \sqrt{x}$ with positively oriented boundary C (i.e. counter clockwise). Evaluate

$$\oint_C \left[9x^2y(x^3+1)^{\frac{1}{2}} - xy^2(x^3+1)^{\frac{3}{2}}\right] dx + \left[2(x^3+1)^{\frac{3}{2}} + 2(y^3+1)^{\frac{3}{2}}\right] dy.$$

Sol.



Let $P(x,y) = 9x^2y(x^3+1)^{\frac{1}{2}} - xy^2(x^3+1)^{\frac{3}{2}}$ and $Q(x,y) = 2(x^3+1)^{\frac{3}{2}} + 2(y^3+1)^{\frac{3}{2}}$

$$\frac{\partial Q}{\partial x} = 9x^2(x^2+1)^{\frac{1}{2}}, \qquad \frac{\partial P}{\partial y} = 9x^2(x^2+1)^{\frac{1}{2}} - 2xy(x^3+1)^{\frac{3}{2}}$$

By Green Theorem:

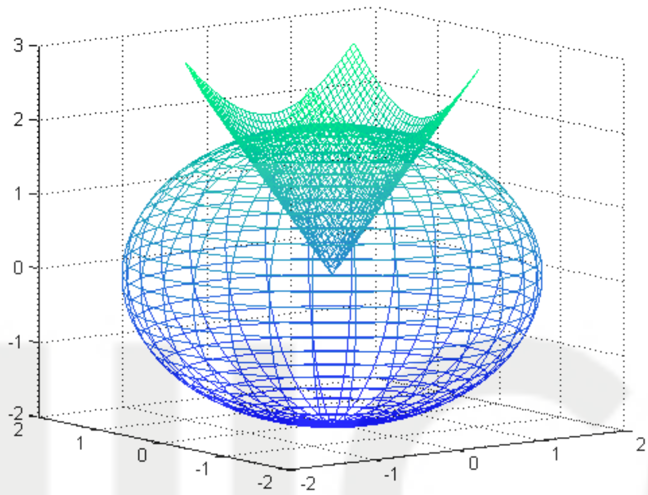
$$\begin{aligned}
&\oint_C Pdx + Qdy \\
&= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \int_0^1 \int_0^{\sqrt{x}} 2xy(x^3+1)^{\frac{3}{2}} dy dx \\
&= \int_0^1 x^2(x^3+1)^{\frac{3}{2}} dx \\
&= \frac{2}{15}(x^3+1)^{\frac{5}{2}}\Big|_0^1 \\
&= \frac{2}{15}(2^{\frac{5}{2}} - 1)
\end{aligned}$$

□

4. Evaluate the triple integral $\iiint_E xyz\, dV$ with

$$E = \left\{0 \le x \le \sqrt{4-y^2},\ 0 \le y \le 2,\ \sqrt{x^2+y^2} \le z \le \sqrt{8-x^2-y^2}\right\}.$$

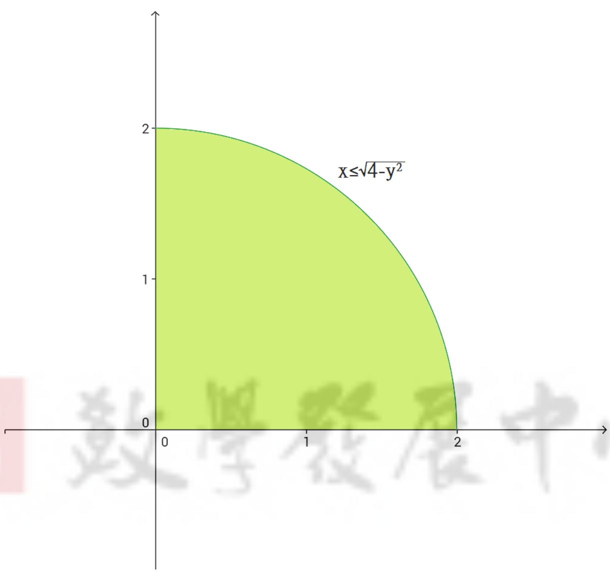
Sol.



(Method 1) Integrate directly.

$$\iiint_E xyz\, dV = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} xyz\, dz dx dy$$

(Method 2) Integrate in cylindrical coordinate.



$$\iiint_E xyz\ dV = \int_0^{\frac{\pi}{2}}\int_0^2\int_r^{\sqrt{8-r^2}} (r\cos\theta)(r\sin\theta)z\ rdzdrd\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_0^2 (4-r^2)r^3\cos\theta\sin\theta\ drd\theta$$

(Method 3) Integrate in polar coordinate.

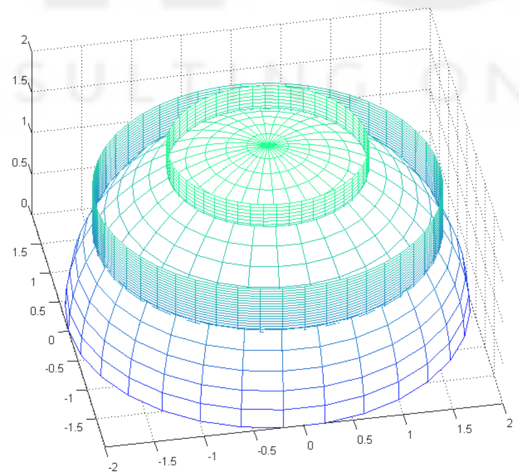
$$\begin{cases} \sqrt{x^2+y^2} = z \\ \sqrt{8-x^2-y^2} = z \end{cases} \Rightarrow \begin{cases} \rho\sin\phi = \rho\cos\phi \\ 8-\rho^2\sin^2\phi = \rho^2\cos^2\phi \end{cases}$$
$$\Rightarrow \begin{cases} \phi = \dfrac{\pi}{4} \\ \rho = 2\sqrt{2} \end{cases}$$

$$\iiint_E xyz\ dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{2\sqrt{2}} (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\ \rho^2 d\rho d\phi d\theta$$

The value of the integral is $\frac{8}{3}$. □

5. Find the area of the surface $\{x^2+y^2+z^2=4,\ 1\le x^2+y^2\le 3,\ z\ge 0\}$.

Sol.

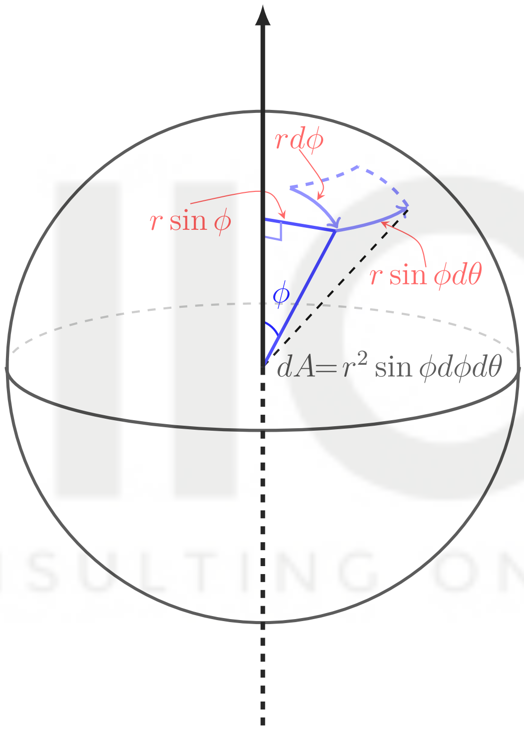


(Method 1)

$$A(S) = \iint_S dS$$
$$= \iint_{1\le x^2+y^2\le 3} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA \quad (z=\sqrt{4-x^2-y^2})$$

$$= \iint_{1\le x^2+y^2\le 3} \sqrt{1+\left(\frac{y^2}{4-x^2-y^2}\right)+\left(\frac{x^2}{4-x^2-y^2}\right)}dA$$
$$= \iint_{1\le x^2+y^2\le 3} \frac{2}{\sqrt{4-x^2-y^2}}dA$$
$$= \int_0^{2\pi}\int_1^{\sqrt{3}} \frac{2}{\sqrt{4-r^2}}rdrd\theta$$
$$= 4\pi(\sqrt{3}-1)$$

(Method 2)



$$A(S) = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} r^2\sin\phi \; d\phi d\theta$$
$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\sin\phi \; d\phi d\theta$$
$$= 4\pi(\sqrt{3}-1)$$

□

6. Let S be the part of the sphere $x^2+y^2+(z-2)^2=8$ that lies above the xy-plane and that has outward normal (i.e. with $\mathbf{k}$-component ≥ 0). Let $\mathbf{F}(x,y,z)=\langle -y^3\cos xz, x^3e^{yz}, -e^{xyz}\rangle$. Find $\displaystyle\iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S}$.

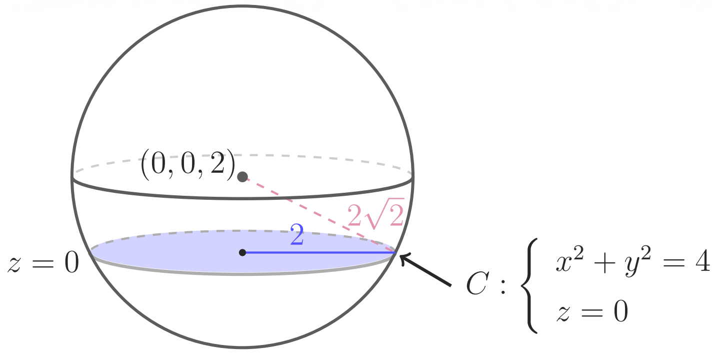
Sol.

(Method 1)

$$\begin{aligned}
\mathbf{r}(t) &= \langle 2\cos t, 2\sin t, 0\rangle \\
\mathbf{r}'(t) &= \langle -2\sin t, 2\cos t, 0\rangle \\
d\mathbf{r} &= \frac{d\mathbf{r}}{dt}dt, \quad 0\leq t\leq 2\pi \\
\iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S} &= \oint_C \mathbf{F}\cdot d\mathbf{r} \\
&= \oint_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt \\
&= \oint_C (-8\sin^3 t, 8\cos^3 t, -1)\cdot(-2\sin t, 2\cos t, 0)dt \\
&= 16\int_0^{2\pi}(\sin^4 t+\cos^4 t)dt = 24\pi
\end{aligned}$$

(Method 2)

$$S:\begin{cases} x^2+y^2(z-2)^2=8 \\ z\geq 0\end{cases} \qquad D:\begin{cases} x^2+y^2\leq 4 \\ z=0\end{cases}$$



$$\begin{aligned}
\iint_S \text{curl } \mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_D \text{curl } \mathbf{F}\cdot d\mathbf{D} \\
&= \iint_D \text{curl } \mathbf{F}\cdot(0,0,1)dS \\
&= \iint_D \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{F}_x & \mathrm{F}_y & \mathrm{F}_z \end{vmatrix}\cdot(0,0,1)dD
\end{aligned}$$

$$
\begin{aligned}
&= \iint_D \left(\frac{\partial}{\partial x}(x^3 e^{yz}) \Big|_{z=0} \right) - \left(\frac{\partial}{\partial y}(-y^3 \cos xz) \Big|_{z=0} \right) dD \\
&= \iint_D (3x^2 + 3y^2) dD \\
&= \int_0^{2\pi} \int_0^2 3r^2 \; r dr d\theta = 24\pi
\end{aligned}
$$

□

7. (a) Find a scalar function $f(x, y, z)$ such that $\nabla f = \sin y \ \mathbf{i} + x \cos y \ \mathbf{j} - \sin z \ \mathbf{k}$.

   (b) Find the line integral $\int_C \sin y dx + x \cos y dy + (y - \sin z) dz$, where $C : \mathbf{r}(t) = \left\langle t, \frac{\pi}{2} \cos t, \frac{\pi}{2} \sin t \right\rangle$, $0 \le t \le \pi$.

Sol.

(a)

$$
\begin{aligned}
\nabla f &= \sin y \ \mathbf{i} + x \cos y \ \mathbf{j} - \sin z \ \mathbf{k} \\
&\Rightarrow f = x \sin y + h(y, z) \\
\frac{\partial f}{\partial y} &= x \cos y + \frac{\partial}{\partial y} h(y, z) = x \cos y \\
&\Rightarrow h(y, z) = C g(z) = g(z) \quad \text{(We let } C = 1) \\
&\Rightarrow f = x \sin y + g(z) \\
\frac{\partial f}{\partial z} &= g'(z) = -\sin z \quad \Rightarrow \ g(z) = \cos z + C
\end{aligned}
$$

$$
\therefore f = x \sin y + \cos z + C
$$

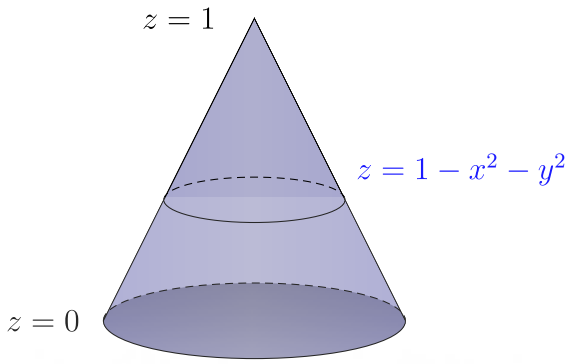
(b)

$$
\begin{aligned}
& \int_C \sin y dx + x \cos y dy + (y - \sin z) dz \\
=& \int \nabla f \cdot d\mathbf{r} + \int y dz \\
=& \ [f(\mathbf{r}(\pi)) - f(\mathbf{r}(0))] + \int \frac{\pi}{2} \cos t \cdot (\frac{\pi}{2} \cos t \ dt) \\
=& \ f(\pi, -\frac{\pi}{2}, 0) - f(0, \frac{\pi}{2}, 0) + \frac{\pi^2}{4} \int_0^\pi \frac{1 + \cos 2t}{2} dt \\
=& \ -\pi + \left(\frac{\pi}{2}\right)^3
\end{aligned}
$$

□

8. Let $\mathbf{F}=\langle 3xy^2, y^3, e^{x^2+y^2}\rangle$. Let S be the part of the surface $z = 1-x^2-y^2$ that lies above xy-plane oriented upwards(that is, with normal having $\mathbf{k}$-component ≥ 0). Calculate the flux $\int_S \mathbf{F}\cdot dS$ of $\mathbf{F}$ across S. Note that S is not closed.

Sol.



(Method 1)

$$\int_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(r(x,y))\cdot(r_x \times r_y)dA$$

$$r(x,y) = (x, y, 1-x^2-y^2)$$
$$r_x \times r_y = (1,0,-2x)\times(0,1,-2y) = (2x, 2y, 1)$$

$$\begin{aligned}\therefore \int_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D 6x^2y^2 + 2y^4 + e^{x^2+y^2}dA \\ &= \int_0^{2\pi}\int_0^1 \left(6(r\cos\theta)^2(r\sin\theta)^2 + 2(r\sin\theta)^4 + e^{r^2}\right) rdrd\theta \\ &= \pi\left(e - \frac{1}{2}\right)\end{aligned}$$

(Method 2) Let S_1 be the lateral wall of S, and S_2 be the bottom surface of S.

$$\iiint \nabla\cdot\mathbf{F}dV = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S}_1 + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}_2$$

$$\begin{aligned}&\therefore \iint_{S_1}\mathbf{F}\cdot d\mathbf{S}_1 \\ &= \iiint(\nabla\cdot\mathbf{F})dV - \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}_2 \\ &= \iiint 6y^2 dV - \iint \mathbf{F}\cdot(0,0,-1)d\mathbf{S}_2\end{aligned}$$

$$
\begin{aligned}
&= \iiint 6y^2 dV - \iint_D -e^{x^2+y^2} dA \\
&= \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 6(r\sin\theta)^2 r dz dr d\theta + \int_0^{2\pi} \int_0^1 e^{r^2} r dr d\theta \\
&= \pi\left(e - \frac{1}{2}\right)
\end{aligned}
$$

□