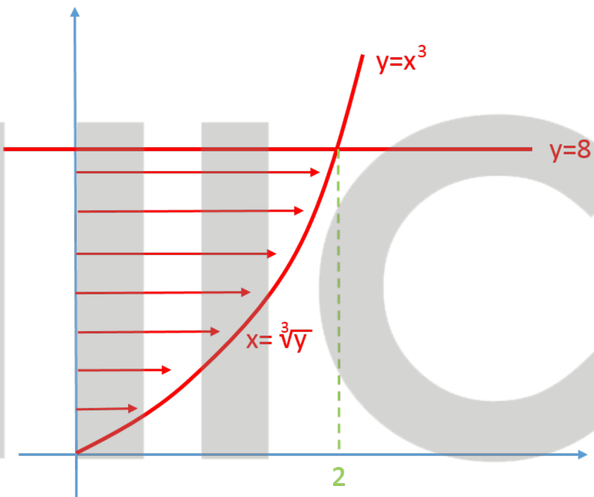
101-2 微甲 01-04 班期末考試題及詳解

1. Evaluate $\displaystyle\int_0^2 \int_{x^3}^8 \frac{x^5}{\sqrt{x^6+y^2}}dydx.$

Sol.

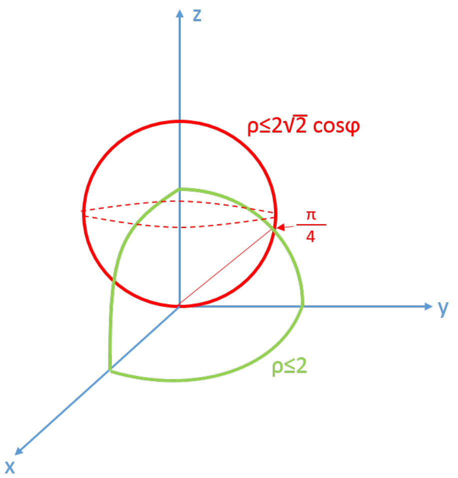


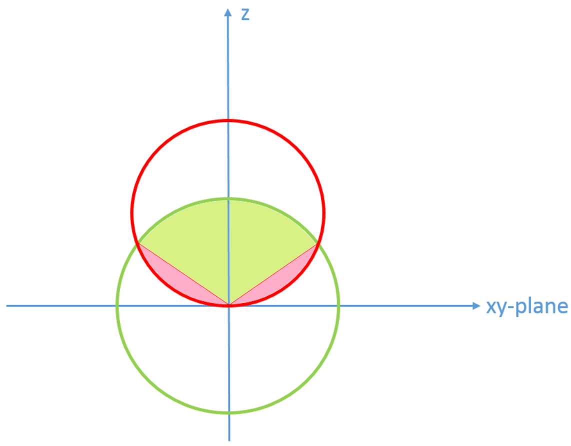
$$
\begin{aligned}
&\int_0^2 \int_{x^3}^8 \frac{x^5}{\sqrt{x^6+y^2}}dydx \\
&= \int_0^8 \int_0^{\sqrt[3]{y}} \frac{x^5}{\sqrt{x^6+y^2}}dxdy \\
&= \int_0^8 \sqrt{x^6+y^2}\cdot \frac{1}{6\cdot\frac{1}{2}}\Big|_0^{\sqrt[3]{y}} dy \\
&= \frac{1}{3}\int_0^8 (\sqrt{2}-1)ydy \\
&= \frac{1}{3}(\sqrt{2}-1)\frac{y^2}{2}\Big|_0^8 \\
&= \frac{32}{3}(\sqrt{2}-1)
\end{aligned}
$$

□

2. Find the volume of the solid common to the balls $\rho \leq 2\sqrt{2}\cos\phi$ and $\rho \leq 2$.

Sol.





$$
\begin{aligned}
&\rho \leq 2\sqrt{2}\cos\phi \\
\Rightarrow\ &(x^2+y^2+z^2)^{\frac{1}{2}} \leq 2\sqrt{2}\frac{z}{(x^2+y^2+z^2)^{\frac{1}{2}}} \\
\Rightarrow\ &x^2+y^2+(z-\sqrt{2})^2 \leq 2 \\
&\rho \leq 2 \\
\Rightarrow\ &x^2+y^2+z^2 \leq 4 \\
&\text{when } 2\sqrt{2}\cos\phi = 2 \Rightarrow \phi = \frac{\pi}{4}
\end{aligned}
$$

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^2 \rho^2\sin\phi\, d\rho d\phi d\theta + \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\sqrt{2}\cos\phi} \rho^2\sin\phi\, d\rho d\phi d\theta \\
&= 2\pi(-\cos\phi)\Big|_0^{\frac{\pi}{4}} \cdot \frac{\rho^3}{3}\Big|_0^2 + 2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi\frac{(2\sqrt{2}\cos\phi)^3}{3}d\phi \\
&= \frac{2\pi}{3}(8-4\sqrt{2}) + \frac{32\sqrt{2}\pi}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\phi\cos^3\phi\, d\phi \\
&= \frac{2\pi}{3}(8-4\sqrt{2}) + \frac{32\sqrt{2}\pi}{3}\cdot\left(-\frac{1}{4}\cos^4\phi\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \frac{16}{3}\pi - 2\sqrt{2}\pi
\end{aligned}
$$

□

3. Evaluate the integral $\iint_\Omega \sin(3x^2-2xy+3y^2)dxdy$, where Ω is the ellipse $3x^2-2xy+3y^2 \leq 2$. You may try the change of variables $x = u+kv,\ y = u-kv$ for some constant k.

Sol.

$$\text{Let } x = u+kv,\ y = u-kv$$

$$\begin{aligned}
3x^2-2xy+3y^2 &= 3(2u^2+2k^2v^2)-2(u^2-k^2v^2)\\
&= 4u^2+8k^2v^2 \leq 2 \quad \text{let } k^2 = \frac{1}{2} \to k = \frac{1}{\sqrt{2}}\\
&\Rightarrow 4u^2+4v^2 \leq 2\\
&\Rightarrow u^2+v^2 \leq \frac{1}{2}
\end{aligned}$$

$$J = \left\| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right\| = \left\| \det \begin{pmatrix} 1 & k \\ 1 & -k \end{pmatrix} \right\| = |-2k| = \sqrt{2}$$

$$\begin{aligned}
&\iint_\Omega \sin(3x^2-2xy+3y^2)dxdy\\
=& \iint_{u^2+v^2\leq\frac{1}{2}} \sin[4(u^2+v^2)]\sqrt{2}dudv\\
=& \sqrt{2}\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} \sin(4r^2)rdrd\theta\\
=& 2\pi\sqrt{2}\frac{-1}{8}\cos(4r^2)\Big|_0^{\frac{1}{\sqrt{2}}}\\
=& \frac{\sqrt{2}\pi}{4}(1-\cos 2)
\end{aligned}$$

□

4. For $y > 0$, let

$$\mathbf{F}(x,y,z) = (e^{-x}\ln y - z)\mathbf{i} + (2yz - \frac{e^{-x}}{y})\mathbf{j} + (y^2 - x)\mathbf{k}$$
$$\mathbf{G}(x,y,z) = e^{-x}\ln y\,\mathbf{i} + (2yz - \frac{e^{-x}}{y})\mathbf{j} - x\,\mathbf{k}$$

(a) Show that the vector function $\mathbf{F}$ is a gradient on $\{(x,y,z) \mid y > 0\}$ by finding an f such that $\nabla f = \mathbf{F}$.

(b) Evaluate the line integral $\int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(u) = (1 + u^2)\mathbf{i} + e^u\mathbf{j} + (1+u)\mathbf{k},\ u \in [0,1]$.

Sol.

(a)

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

$$\begin{aligned}\frac{\partial f}{\partial x} &= e^{-x}\ln y - z \\ \Rightarrow f &= -e^{-x}\ln y - zx + g(y,z)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{-e^{-x}}{y} + \frac{\partial g(y,z)}{\partial y} = 2yz - \frac{e^{-x}}{y} \\ &\therefore g(y,z) = y^2 z + h(z) \\ \Rightarrow f &= -e^{-x}\ln y - zx + y^2 z + h(z)\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= -x + y^2 + \frac{dh(z)}{dz} = y^2 - x \\ &\therefore h(z) = C \\ \Rightarrow f &= -e^{-x}\ln y - zx + y^2 z + C\end{aligned}$$

(b)

$$\begin{aligned}&\oint \mathbf{G}(r) \cdot d\mathbf{r} \\ =& \oint [\nabla f + (z\mathbf{i} - y^2\mathbf{k})] \cdot d\mathbf{r} \\ =& f(2,e,2) - f(1,1,1) + \int_0^1 (1+u)2u du - \int_0^1 e^{2u} du \\ =& \frac{3}{2}e^2 - e^{-2} - \frac{11}{6}\end{aligned}$$

□

5. Let C be a piecewise-smooth Jordan curve that does not pass through the origin. Evaluate $\oint_C \frac{-y^5}{(x^2+y^2)^3}dx + \frac{xy^4}{(x^2+y^2)^3}dy$ for the following two cases, where C is traversed in the counter-clockwise direction.

(a) C does not enclose the origin.

(b) C does enclose the origin.

Sol.

(a) Let $P = \frac{-y^5}{(x^2+y^2)^3}$ and $Q = \frac{xy^4}{(x^2+y^2)^3}$

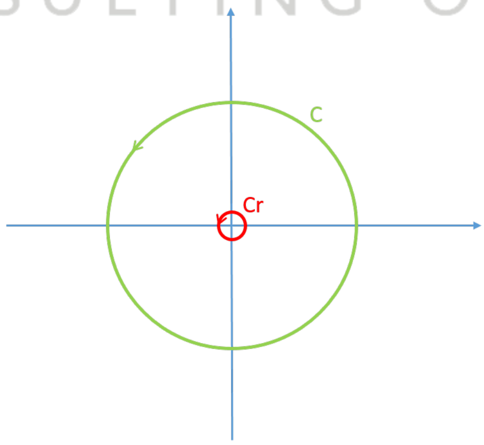
By Green's theorem: $\oint_C Pdx + Qdy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$

$$\iint_\Omega \left(\frac{y^6 - 5x^2y^4}{(x^2+y^2)^4} - \frac{y^6 - 5x^2y^4}{(x^2+y^2)^4}\right) dxdy = 0$$

(b)

$$\begin{aligned}
&\oint_C Pdx + Qdy - \oint_{C_r} Pdx + Qdy \\
&= \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = 0 \\
&\therefore \oint_C Pdx + Qdy = \oint_{C_r} Pdx + Qdy
\end{aligned}$$

$C_r:\ x = \epsilon\cos\theta,\ y = \epsilon\sin\theta$



$$\begin{aligned}
\oint_{C_r} Pdx + Qdy &= \int_0^{2\pi} \frac{-(\epsilon\sin\theta)^5}{\epsilon^6}(-\epsilon\sin\theta) + \frac{\epsilon^5\cos\theta\sin^4\theta}{\epsilon^6}\epsilon\cos\theta\ d\theta \\
&= \int_0^{2\pi} (\sin^6\theta + \cos^2\theta\sin^4\theta)d\theta
\end{aligned}$$

$$= \int_0^{2\pi} \sin^4 \theta d\theta = \int_0^{2\pi} (\frac{1-\cos 2\theta}{2})^2 d\theta$$
$$= \int_0^{2\pi} \frac{1}{4} - \frac{1}{2}\cos 2\theta + \frac{1}{8}(1+\cos 4\theta) d\theta$$
$$= \frac{3\pi}{4}$$

□

6. Let S be the triangular region with vertices $(0,0,0)$, $(a,0,0)$ and (a,a,a), $a>0$, with upward unit normal $\mathbf{n}$, and C be the positively oriented boundary of S. Let

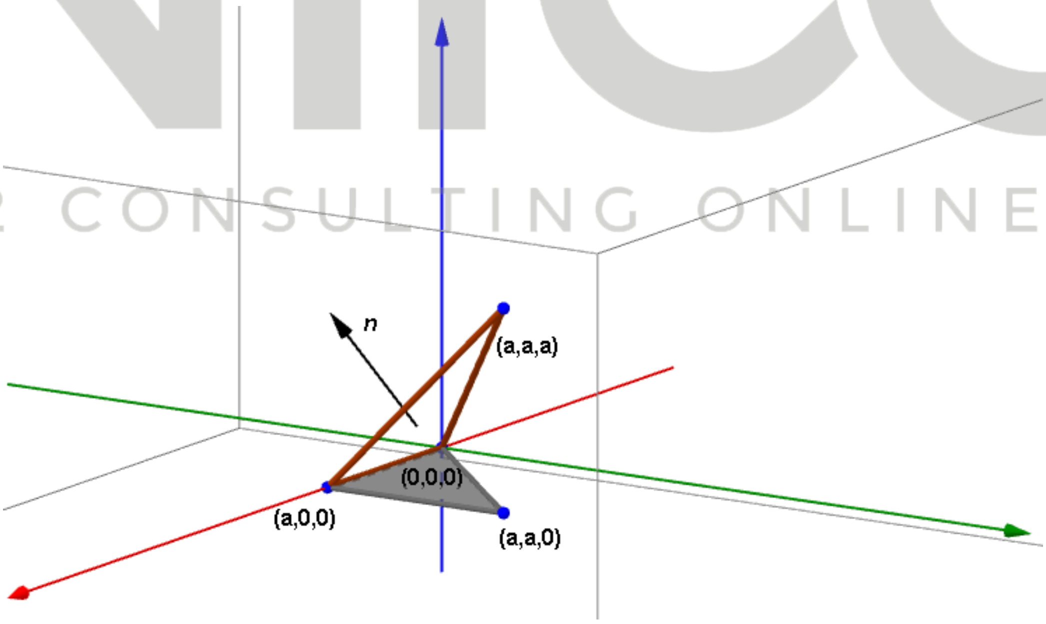
$$\mathbf{F} = (y - z\cos(x^2))\mathbf{i} + (2x - \sin(z^2))\mathbf{j} + (3z - \tan(y^2))\mathbf{k}$$

   (a) Find a parametrization of S and find the upward unit normal $\mathbf{n}$. (Hint: consider the projection of S to xy-plane.)

   (b) Evaluate $\nabla \times \mathbf{F}$.

   (c) Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Sol.



(a)

$$(a,0,0) \times (a,a,a) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & 0 & 0 \\ a & a & a \end{vmatrix} = -a^2\mathbf{j} + a^2\mathbf{k}$$

$$\therefore \mathbf{n} = \frac{1}{\sqrt{2}}(0, -1, 1)$$

$$S : \{(x, y, y) \mid 0 \leq y \leq x \leq a\} \quad (\because (x, y, y) \cdot \mathbf{n} = 0)$$

(b)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = [2z\cos(z^2) - 2y\sec(y^2)]\mathbf{i} - \cos(x^2)\mathbf{j} + \mathbf{k}$$

(c)

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dA \\ &= \iint_S \frac{1}{\sqrt{2}}(1 + \cos x^2) dA \\ &= \iint_{S'(xy)} \frac{1}{\sqrt{2}}(1 + \cos x^2)|\mathbf{r}_x \times \mathbf{r}_y| dy dx \\ &= \int_0^a \int_0^x (1 + \cos x^2) dy dx \\ &= \frac{1}{2}(a^2 + \sin a^2) \end{aligned}$$

Note that:

$$\begin{aligned} z &= f(x, y) = y \\ \mathbf{r}_x &= \mathbf{i} + (\frac{\partial f}{\partial x})\mathbf{k} \\ \mathbf{r}_y &= \mathbf{j} + (\frac{\partial f}{\partial y})\mathbf{k} \end{aligned}$$

$$\mathbf{r}_x \times \mathbf{r}_y \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -\mathbf{j} + \mathbf{k}$$

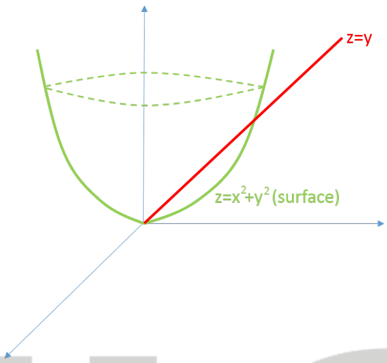
□

7. Let S_1 be the surface $\{(x,y,z) \mid z = x^2+y^2,\ z \le y\}$, S_2 be the surface $\{(x,y,z) \mid z = y,\ x^2+y^2 \le z\}$, and $\mathbf{V}(x,y,z) = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$.

(a) Compute directly the downward flux of $\mathbf{V}$ across S_1

(b) Use the divergence theorem to compute the upward flux of $\mathbf{V}$ across S_2.

Sol.



(a) $S_1 : (x, y, x^2+y^2),\ x^2+y^2 \le y$

$$\begin{aligned} \mathbf{r}_x &= (1,0,2x) \\ \mathbf{r}_y &= (0,1,2y) \end{aligned} \qquad \mathbf{r}_x \times \mathbf{r}_y = (-2x,-2y,1)$$

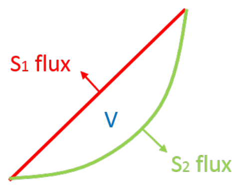
$$\begin{aligned} \therefore \iint_{S_1} V \cdot dS_1 &= \iint_{S_1'} V \cdot (-\mathbf{r}_x \times \mathbf{r}_y)\,dxdy \\ &= \iint_{S_1'} [-2xy + 2xy - (x^2+y^2)]\,dxdy \\ &= \int_0^{\pi}\int_0^{\sin\theta} -r^2 r\,dr\,d\theta \\ &= -\frac{1}{4}\int_0^{\pi} \sin^4\theta \\ &= \frac{-3}{32}\pi \end{aligned}$$

Note that:

$$\begin{aligned} &r^2 \le r\sin\theta\ (x^2+y^2 \le y) && \Rightarrow r \le \sin\theta \\ &z = x^2+y^2 \ge 0,\ z \le y && \Rightarrow y \ge 0,\ 0 < \theta < \pi \end{aligned}$$

(b)

$$\iiint_{\Omega} (\nabla \cdot V)\,d\Omega(\text{volume}) = \iint_{S_1} V \cdot dA + \iint_{S_2} V \cdot dA$$



$$\iiint_{\Omega} (\nabla \cdot V) d\Omega = \iiint 1 \cdot d\Omega$$
$$= \int_0^{\pi} \int_0^{\sin\theta} \int_{r^2}^{r\sin\theta} r dz dr d\theta$$
$$= \int_0^{\pi} \int_0^{\sin\theta} (r^2 \sin\theta - r^3) dr d\theta$$
$$= \frac{\pi}{32}$$

$$\therefore \iint_{S_2} V \cdot dA = \frac{\pi}{32} - \frac{-3}{32}\pi = \frac{\pi}{8}$$

Note that:

$$z_{max} = y = r\sin\theta \qquad z_{min} = x^2 + y^2 = r^2$$
$$r^2 \leq r\sin\theta \qquad \Rightarrow r \leq \sin\theta$$

□